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FURTHER DEVELOPMENT OF THE RELATIONS BETWEEN IMPULSIVE SCREWS AND INSTANTANEOUS SCREWS, BEING THE ELEVENTH MEMOIR ON THE "THEORY OF SCREWS."

By SIR ROBERT S. BALL, LL.D., F.R.S., Lowndean Professor of Astronomy and Geometry at Cambridge.

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IV.

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[Read November 9, 1896.]

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I.—On Expressions for the Kinetic Energy and Twist Velocity.

Let us suppose that a body of mass M is twisting around a screw α with the velocity $\dot{\alpha}$. Then it is obvious that the kinetic energy of the body is to be expressed in the form $M\dot{\alpha}^2$ multiplied into some expression which has the dimensions of the square of a linear magnitude. This magnitude has a geometrical significance which is worthy of note.

Let η be the impulsive screw which corresponds to α as an instantaneous screw.

Let p_α represent the pitch of α.

Let $(\alpha\eta)$ represent the angle between α and η.

Let $\varpi_{a\eta}$ denote the virtual coefficient of α and η; then the required expression for the kinetic energy is

$$M\dot{a}^2 \frac{p_a}{\cos(a\eta)} \varpi_{a\eta}.$$

The physical meaning of the expression should be noted. η could not be reciprocal to α, so $\varpi_{a\eta}$ could never vanish. p_a could not vanish without $\cos(a\eta)$ vanishing also.

The following is the proof:

From the known formulæ, "Theory of Screws," p. 47,

$$H\eta_1 = +\, a\alpha_1, \qquad H\eta_2 = -\, a\alpha_2,$$
$$H\eta_3 = +\, b\alpha_3, \qquad H\eta_4 = -\, b\alpha_4,$$
$$H\eta_5 = +\, c\alpha_5, \qquad H\eta_6 = -\, c\alpha_6,$$

whence $H[(\eta_1+\eta_2)(\alpha_1+\alpha_2)+(\eta_3+\eta_4)(\alpha_3+\alpha_4)+(\eta_5+\eta_6)(\alpha_5+\alpha_6)] = p_a,$

or $$H = \frac{p_a}{\cos(a\eta)},$$

so that $$\frac{p_a}{\cos a\eta}\eta_1 = +\, a\alpha_1, \qquad \frac{p_a}{\cos a\eta}\eta_2 = -\, a\alpha_2, \text{ \&c.},$$

whence $$\frac{p_a}{\cos a\eta}(a\alpha_1\eta_1 - a\alpha_2\eta_2 \ldots) = a^2\alpha_1^2 + a^2\alpha_2^2 \ldots,$$

or $$\frac{p_a}{\cos a\eta}\varpi_{a\eta} = a^2\alpha_1^2 + a^2\alpha_2^2 + b^2\alpha_3^2 + b^2\alpha_4^2 + c^2\alpha_5^2 + c^2\alpha_6^2.$$

But the kinetic energy is $M\dot{a}^2$ ($a^2\alpha_1^2$ + &c.), whence we obtain the theorem required.

It is easy to show that, if the unit impulsive wrench be given on a screw η to a quiescent body of unit mass, which thereupon commences to twist about a screw α, the twist velocity acquired is

$$\frac{\cos(a\eta)}{p_a}.$$

These expressions for the twist velocity and the kinetic energy are specially interesting, inasmuch as they introduce the coordinates of the rigid body solely by means of the impulsive screw.

II.—On the General Relations between two Pairs of Impulsive Screws and Instantaneous Screws.

In a recent note communicated to the Cambridge Philosophical Society, I enunciated a theorem of which I am now to give the demonstration. To make the matter clear, it may be convenient to reproduce here what has already appeared in the Proceedings of the Cambridge Philosophical Society, vol. ix., part iii., p. 193.

"Let α be a screw about which a free rigid body is made to twist in consequence of an impulsive wrench administered on some other screw η. Except in the case where α and η are reciprocal, it will always be possible (in many different ways) to design and place a rigid body so that two arbitrarily chosen screws α and η will possess the required relation.

"Let now β and ξ be two other screws (not reciprocal): we may consider the question as to whether a rigid body can be designed and placed so that α shall be the instantaneous screw corresponding to η as an impulsive screw, while β bears the same relation to ξ.

"It is easy to see that it will not generally be possible for α, β, η, ξ to stand in the required relations. For, taking α and β as given, there are five disposable quantities in the choice of η, and five more in the choice of ξ. We ought, therefore, to have ten disposable coordinates for the designing and the placing of the rigid body. But there are not so many. We have three for the coordinates of its centre of gravity, three for the direction of its principal axes, and three more for the radii of gyration. The density of the body is not disposable, the other nine quantities being assigned the relations of the impulsive and instantaneous screws, as geometrical entities are not affected by alterations in the density. This produces its effect in the twist velocity acquired, and not in the position of the screw.

"It thus appears that if the four screws had been chosen arbitrarily we should have ten conditions to satisfy, and only nine disposable coordinates. It is hence plain that the four screws cannot be chosen quite arbitrarily. They must be in some way restricted. The object of the present note is to show that these restrictions are two in number, and to set down what they are.

"Draw a cylindroid A through α, β, and another cylindroid P through $\eta\xi$. Then an impulsive wrench about any screw ω on P will make the body twist about some screw θ on A. As ω moves over P, so will its correspondent θ travel over A. It is easy to show that any four screws on P will be equianharmonic with their four correspondents on A, and that consequently the two systems will be homographic.

"In general, to establish the homography of two cylindroids, three corresponding pairs of screws must be given; and, of course, there could be a triply infinite variety in the possible homographies. It is, however, a somewhat remarkable fact that in the particular homography with which we are concerned there is no arbitrary element. Given the cylindroids A and P, then, without any other considerations whatever, all the corresponding pairs are determined. This is first to be proved.

"If the mass be one unit, and the intensity of the impulsive wrench on ω be one unit, then the twist velocity acquired by θ is

$$\frac{\cos(\theta\omega)}{p_\theta},$$

where $\cos(\theta\omega)$ denotes the cosine of the angle between the two screws θ and ω, and where p_θ is the pitch of θ. If, therefore, p_θ be zero, then $\cos(\theta\omega)$ must be zero. In other words, the two impulsive screws ω_1, ω_2 on P, which correspond to the two screws of zero pitch θ_1, θ_2 on A, must be at right angles to them, respectively. This will at once identify the correspondents on P to two known screws on A.

"We have thus two pairs of correspondents given, and we can now show a third pair. For if ω_3 be a screw on P reciprocal to θ_2, then its correspondent θ_3 will be reciprocal to ω_2. Thus we have three pairs θ_1, θ_2, θ_3 on A, and their three correspondents ω_1, ω_2, ω_3 on P. This establishes the homography, and the correspondent θ to any other screw ω is assigned by the condition that the anharmonic ratio of $\omega_1\omega_2\omega_3\omega$ is the same as that of $\theta_1\theta_2\theta_3\theta$.

"Reverting to our original screws α and η, β and ξ, we now see that they must fulfil the conditions

$$(\omega_1\omega_2\omega_3\eta) = (\theta_1\theta_2\theta_3\alpha), \qquad (\omega_1\omega_2\omega_3\xi) = (\theta_1\theta_2\theta_3\beta)$$

when the quantities in the brackets denote the anharmonic ratios.

"On working out the relations, they assume a very simple form:

$$\frac{p_\alpha}{\cos(\alpha\eta)}\cos(\beta\eta)+\frac{p_\beta}{\cos(\beta\xi)}\cos(\alpha\xi)=2\varpi_{\alpha\beta},$$

$$\frac{p_\alpha}{\cos(\alpha\eta)}\varpi_{\beta\eta}=\frac{p_\beta}{\cos(\beta\xi)}\varpi_{\alpha\xi}.$$

I had already given the second of these equations in the 'Tenth Memoir on the Theory of Screws,' Transactions, Royal Irish Academy, vol. xxx., p. 571. I did not then know that there must be the further condition represented by the first of these equations. Nor was I aware of the unique homography between the screws on two cylindroids which expressed the geometrical conditions under which alone one of these cylindroids could be the locus of impulsive screws corresponding to the screws of the other as instantaneous screws."

We have now to demonstrate the first of the formulæ above given.

Expanding the left-hand side, it becomes

$$+\frac{p_\alpha}{\cos(\alpha\eta)}\left((\beta_1+\beta_2)(\eta_1+\eta_2)+(\beta_3+\beta_4)(\eta_3+\eta_4)+(\beta_5+\beta_6)(\eta_5+\eta_6)\right)$$

$$+\frac{p_\beta}{\cos(\beta\xi)}\left((\alpha_1+\alpha_2)(\xi_1+\xi_2)+(\alpha_3+\alpha_4)(\xi_3+\xi_4)+(\alpha_5+\alpha_6)(\xi_5+\xi_6)\right).$$

But, as already shown,

$$\frac{p_\alpha}{\cos(\alpha\eta)}\eta_1=+a\alpha_1\quad\frac{p_\alpha}{\cos(\alpha\eta)}\eta_2=-a\alpha_2,\ \&c.,$$

$$\frac{p_\beta}{\cos(\beta\xi)}\xi_1=+a\beta_1\quad\frac{p_\beta}{\cos(\beta\xi)}\xi_2=-a\beta_2,\ \&c.;$$

whence, by substitution, the expression reduces to

$$\begin{aligned}&+a(\beta_1+\beta_2)(\alpha_1-\alpha_2)+a(\alpha_1+\alpha_2)(\beta_1-\beta_2)\\&+b(\beta_3+\beta_4)(\alpha_3-\alpha_4)+b(\alpha_3+\alpha_4)(\beta_3-\beta_4)\\&+c(\gamma_3+\gamma_4)(\alpha_5-\alpha_6)+c(\alpha_5+\alpha_6)(\beta_5-\beta_6)\\&=2a\alpha_1\beta_1-2a\alpha_2\beta_2+2b\alpha_3\beta_3-2b\alpha_4\beta_4+2c\alpha_5\beta_5-2c\alpha_6\beta_6\\&=2\varpi_{\alpha\beta}.\end{aligned}$$

It will, however, be instructive to consider this formula in the way in which it was first actually suggested. The argument is as follows:

Let ϵ be the angle between α and β, ω the angle between η and ξ.

Draw the cylindroid (α, β), and choose upon it a screw which makes an angle θ with α.

Let the corresponding screw on the cylindroid (η, ξ) make an angle ϕ with η.

The coordinates of the screw θ on $(\alpha\beta)$ are

$$\alpha_1 \frac{\sin(\epsilon-\theta)}{\sin\epsilon} + \beta_1 \frac{\sin\theta}{\sin\epsilon}, \quad \alpha_2 \frac{\sin(\epsilon-\theta)}{\sin\epsilon} + \beta_2 \frac{\sin\theta}{\sin\epsilon}, \quad \ldots$$

$$\alpha_6 \frac{\sin(\epsilon-\theta)}{\sin\epsilon} + \beta_6 \frac{\sin\theta}{\sin\epsilon};$$

while the coordinates of the corresponding screw on $(\eta\xi)$ are

$$\eta_1 \frac{\sin(\omega-\phi)}{\sin\omega} + \xi_1 \frac{\sin\phi}{\sin\omega}, \quad \eta_2 \frac{\sin(\omega-\phi)}{\sin\omega} + \xi_2 \frac{\sin\phi}{\sin\omega}, \quad \ldots$$

$$\eta_6 \frac{\sin(\omega-\phi)}{\sin\omega} + \xi_6 \frac{\sin\phi}{\sin\omega}.$$

If a screw of the first type represented by θ_0 be reciprocal to η, then the corresponding screw of the second type represented by ϕ_0 will be reciprocal to α, and we have the conditions

$$\frac{\sin(\epsilon-\theta_0)}{\sin\epsilon}\varpi_{\alpha\eta} + \frac{\sin\theta_0}{\sin\epsilon}\varpi_{\beta\eta} = 0,$$

$$\frac{\sin(\omega-\phi_0)}{\sin\omega}\varpi_{\alpha\eta} + \frac{\sin\phi_0}{\sin\omega}\varpi_{\alpha\xi} = 0;$$

whence by division

$$\frac{\sin(\epsilon-\theta_0)}{\sin(\omega-\phi_0)} = \frac{\sin\theta_0}{\sin\phi_0}\frac{\varpi_{\beta\eta}}{\varpi_{\alpha\xi}}.$$

But the fact that θ and ϕ trace out two homographic systems give for every pair of values of θ and ϕ a relation of the kind

$$\frac{\sin(\epsilon-\theta)}{\sin(\omega-\phi)} = \frac{\sin\theta}{\sin\phi}\kappa,$$

in which κ is constant. Combining this with the last, we see that

$$\kappa = \frac{\varpi_{\beta\eta}}{\varpi_{\alpha\xi}},$$

and thus we have for the general equation defining the correspondence

$$\frac{\sin(\epsilon - \theta)}{\sin(\omega - \phi)} = \frac{\sin\theta}{\sin\phi}\frac{\varpi_{\beta\eta}}{\varpi_{\alpha\xi}},$$

which may be written

$$\frac{\sin\theta}{\sin(\epsilon - \theta)}\varpi_{\beta\eta} = \frac{\sin\phi}{\sin(\omega - \phi)}\varpi_{\alpha\xi}.$$

It had been already known (see "Tenth Memoir") that if α, β, and η, ξ, be any two corresponding pairs of impulsive and instantaneous screws, then

$$\frac{\cos(\alpha\eta)}{p_\alpha}\varpi_{\alpha\xi} = \frac{\cos(\beta\xi)}{p_\beta}\varpi_{\beta\eta}.$$

It will be interesting to see how this is verified when, instead of β and ξ, we take the more general screws ρ and σ, typified by

$$\alpha_1\frac{\sin(\epsilon - \theta)}{\sin\epsilon} + \beta_1\frac{\sin\theta}{\sin\epsilon}, \text{ \&c.},$$

$$\eta_1\frac{\sin(\omega - \phi)}{\sin\omega} + \xi_1\frac{\sin\phi}{\sin\omega}, \text{ \&c.}$$

We may first transform the coordinates of the last screw so as to obtain them all expressed in terms of θ. We have

$$\sin\phi = \frac{\sin\theta}{\sin(\epsilon - \theta)}\sin(\omega - \phi)\frac{\varpi_{\beta\eta}}{\varpi_{\alpha\xi}},$$

whence

$$\eta_1\frac{\sin(\omega - \phi)}{\sin\omega} + \xi_1\frac{\sin\phi}{\sin\omega}$$

becomes

$$\eta_1\frac{\sin(\omega - \phi)}{\sin\omega} + \xi_1\frac{\sin\theta}{\sin(\epsilon - \theta)}\frac{\sin(\omega - \phi)}{\sin\omega}\frac{\varpi_{\beta\eta}}{\varpi_{\alpha\xi}}.$$

Discarding all common factors, inasmuch as the equations are homogeneous, we have the following result:

The screw σ, whose coordinates are,

$$\alpha_1 \sin(\epsilon-\theta)+\beta_1 \sin\theta, \quad \alpha_2 \sin(\epsilon-\theta)+\beta_2 \sin\theta, \text{ \&c.},$$

has, as its corresponding impulsive screw ρ,

$$\eta_1 \sin(\epsilon-\theta)\,\varpi_{\alpha\xi}+\xi_1 \sin\theta\varpi_{\beta\eta}, \qquad \eta_2 \sin(\epsilon-\theta)\,\varpi_{\alpha\xi}+\xi_2 \sin\theta\varpi_{\beta\eta}, \text{ \&c.}$$

These expressions exhibit the correspondence as clearly as it is possible, so long as the perfect generality of the problem is preserved.

We wish now to verify that

$$\frac{\cos(\alpha\eta)}{p_\alpha}\varpi_{\alpha\rho}=\frac{\cos(\sigma\rho)}{p_\sigma}\varpi_{\sigma\eta},$$

or

$$\cos(\alpha\eta)\,p_\sigma\,\varpi_{\alpha\rho}=p_\alpha\cos(\sigma\rho)\,\varpi_{\sigma\eta}.$$

Omitting unnecessary factors, we have

$$\varpi_{\alpha\rho}=\sin(\epsilon-\theta)\,\varpi_{\alpha\xi}\varpi_{\alpha\eta}+\sin\theta\varpi_{\alpha\xi}\varpi_{\beta\eta},$$

$$p_\sigma=\sin^2(\epsilon-\theta)p_\alpha+\sin^2\theta p_\beta+2\sin\theta\sin(\epsilon-\theta)\varpi_{\alpha\beta},$$

$$\begin{aligned}\cos(\sigma\rho)=&+\left((\alpha_1+\alpha_2)\sin(\epsilon-\theta)+(\beta_1+\beta_2)\sin\theta\right)\\&\quad\times\left((\eta_1+\eta_2)\sin(\epsilon-\theta)\,\varpi_{\alpha\xi}+(\xi_1+\xi_2)\sin\theta\varpi_{\beta\eta}\right)\\&+\left((\alpha_3+\alpha_4)\sin(\epsilon-\theta)+(\beta_3+\beta_4)\sin\theta\right)\\&\quad\times\left((\eta_3+\eta_4)\sin(\epsilon-\theta)\,\varpi_{\alpha\xi}+(\xi_3+\xi_4)\sin\theta\varpi_{\beta\eta}\right)\\&+\left((\alpha_5+\alpha_6)\sin(\epsilon-\theta)+(\beta_5+\beta_6)\sin\theta\right)\\&\quad\times\left((\beta_5+\eta_6)\sin(\epsilon-\theta)\,\varpi_{\alpha\xi}+(\xi_5+\xi_6)\sin\theta\varpi_{\beta\eta}\right)\\=&\sin^2(\epsilon-\theta)\,\varpi_{\alpha\xi}\cos(\alpha\eta)+\sin\theta\sin(\epsilon-\theta)\,\varpi_{\alpha\xi}\cos(\beta\eta)\\&\quad+\sin\theta\sin(\epsilon-\theta)\,\varpi_{\beta\eta}\cos(\alpha\xi)+\sin^2\theta\varpi_{\beta\eta}\cos(\beta\xi),\end{aligned}$$

$$\varpi_{\sigma\eta}=\sin(\epsilon-\theta)\,\varpi_{\alpha\eta}+\sin\theta\varpi_{\beta\eta}.$$

Substituting, we find

$$\cos(\alpha\eta)(\sin^2(\epsilon-\theta)\,p_\alpha + \sin^2\theta p_\beta + 2\sin\theta\sin(\epsilon-\theta)\,\varpi_{\alpha\beta})$$
$$\times(\sin(\epsilon-\theta)\,\varpi_{\alpha\xi}\,\varpi_{\alpha\eta} + \sin\theta\varpi_{\alpha\xi}\,\varpi_{\beta\eta})$$
$$= p_\alpha(\sin^2(\epsilon-\theta)\,\varpi_{\alpha\xi}\cos(\alpha\eta) + \sin\theta\sin(\epsilon-\theta)\,\varpi_{\alpha\xi}\cos(\beta\eta)$$
$$+ \sin\theta\sin(\epsilon-\theta)\,\varpi_{\beta\eta}\cos(\alpha\xi) + \sin^2\theta\varpi_{\beta\eta}\cos(\beta\xi))$$
$$\times(\sin(\epsilon-\theta)\,\varpi_{\alpha\eta} + \sin\theta\varpi_{\beta\eta}).$$

Multiplying out, we obtain

$$\sin^3(\epsilon-\theta)(p_\alpha\cos(\alpha\eta)\,\varpi_{\alpha\xi}\,\varpi_{\alpha\eta} - p_\alpha\cos(\alpha\eta)\,\varpi_{\alpha\xi}\,\varpi_{\alpha\eta})$$
$$\sin^2(\epsilon-\theta)\sin\theta(p_\alpha\cos(\alpha\eta)\,\varpi_{\alpha\xi}\,\varpi_{\beta\eta} + 2\cos(\alpha\eta)\,\varpi_{\alpha\beta}\,\varpi_{\alpha\xi}\,\varpi_{\alpha\eta}$$
$$- p_\alpha\cos(\alpha\eta)\,\varpi_{\alpha\xi}\,\varpi_{\beta\eta} - p_\alpha\,\varpi_{\alpha\eta}\,\varpi_{\alpha\xi}\cos(\beta\eta) - p_\alpha\,\varpi_{\beta\eta}\,\varpi_{\alpha\eta}\cos(\alpha\xi))$$
$$+ \sin^2\theta\sin(\epsilon-\theta)(p_\beta\cos(\alpha\eta)\,\varpi_{\alpha\xi}\,\varpi_{\alpha\eta} + 2\cos(\alpha\eta)\,\varpi_{\alpha\beta}\,\varpi_{\alpha\xi}\,\varpi_{\beta\eta}$$
$$- p_\alpha\cos(\beta\eta)\,\varpi_{\alpha\xi}\,\varpi_{\beta\eta} - p_\alpha\cos(\alpha\xi)\,\varpi_{\beta\eta}\,\varpi_{\beta\eta} - p_\alpha\,\varpi_{\beta\eta}\,\varpi_{\alpha\eta}\cos(\beta\xi))$$
$$+ \sin^3\theta(p_\beta\cos(\alpha\eta)\,\varpi_{\alpha\xi}\,\varpi_{\beta\eta} - p_\alpha\,\varpi_{\beta\eta}\,\varpi_{\beta\eta}\cos(\beta\xi)).$$

Remembering that $0 = p_\beta\cos(\alpha\eta)\,\varpi_{\alpha\xi} - p_\alpha\cos(\beta\xi)\,\varpi_{\beta\eta}$, we are conducted to the result that if α, β be instantaneous screws, and η, ξ the corresponding impulsive screws, then

$$2\cos(\alpha\eta)\,\varpi_{\alpha\beta}\,\varpi_{\alpha\xi} - p_\alpha\,\varpi_{\alpha\xi}\cos(\beta\eta) - p_\alpha\,\varpi_{\beta\eta}\cos(\alpha\xi) = 0,$$

and similarly

$$2\cos(\beta\xi)\,\varpi_{\alpha\beta}\,\varpi_{\beta\eta} - p_\beta\,\varpi_{\beta\eta}\cos(\alpha\xi) - p_\beta\,\varpi_{\alpha\xi}\cos(\beta\eta) = 0:$$

these equations may be written

$$p_\alpha(\varpi_{\alpha\xi}\cos(\beta\eta) + \varpi_{\beta\eta}\cos(\alpha\xi)) = 2\cos(\alpha\eta)\,\varpi_\alpha\,\varpi_{\alpha\,;}$$

$$p_\beta(\varpi_{\alpha\xi}\cos(\beta\eta) + \varpi_{\beta\eta}\cos(\alpha\xi)) = 2\cos(\beta\xi)\,\varpi_{\alpha\beta}\,\varpi_{\beta\eta}$$

Dividing one by the other, we get

$$\frac{p_\alpha}{p_\beta} = \frac{\cos(\alpha\eta)\,\varpi_{\alpha\xi}}{\cos(\beta\xi)\,\varpi_{\beta\eta}},$$

which is, of course, the known result;

$$\frac{\cos \alpha\eta}{p_\alpha} \varpi_{\alpha\xi} = \frac{\cos \beta\xi}{p_\beta} \varpi_{\beta\eta},$$

while, by eliminating $\varpi_{\alpha\xi}$ and $\varpi_{\beta\eta}$, we obtain

$$\frac{p_\alpha}{\cos(\alpha\eta)} \cos(\beta\eta) + \frac{p_\beta}{\cos(\beta\xi)} \cos(\alpha\xi) = 2\varpi_{\alpha\beta}.$$

The following investigation may be also noted:

Let λ be an impulsive screw situated anywhere, and let μ be the corresponding instantaneous screw.

Let ρ be the impulsive screw on thecylindroid $(\eta\xi)$, and σ the instantaneous screw on the cylindroid (α, β); I propose then to verify the following formula for every corresponding pair of screws, ρ, σ, on the two cylindroids:

$$\frac{\cos(\lambda\mu)}{p_\mu} \varpi_{\mu\rho} = \frac{\cos(\rho\sigma)}{p_\sigma} \varpi_{\sigma\lambda},$$

or rather $$\cos(\lambda\mu)\, \varpi_{\mu\rho}\, p_\sigma = \cos(\rho\sigma)\, \varpi_{\sigma\lambda} p_\mu.$$

As we are dealing with homogeneous equations we may take coordinates as follows:

of σ $\quad \alpha_1 \sin(\epsilon - \theta) + \beta_1 \sin\theta, \qquad \alpha_2 \sin(\epsilon - \theta) + \beta_2 \sin\theta$, &c.,

of ρ $\quad \eta_1 \sin(\epsilon - \theta)\, \varpi_{\alpha\xi} + \xi_1 \sin\theta \varpi_{\beta\eta}, \quad \eta_2 \sin(\epsilon - \theta)\, \varpi_{\alpha\xi} + \xi_2 \sin\theta \varpi_{\beta\eta}$, &c.,

$$\varpi_{\mu\rho} = \sin(\epsilon - \theta)\, \varpi_{\alpha\xi}\, \varpi_{\mu\eta} + \sin\theta \varpi_{\beta\eta}\, \varpi_{\mu\xi},$$

$$p_\sigma = \sin^2(\epsilon - \theta) p_\alpha + \sin^2\theta p_\beta + 2 \sin\theta \sin(\epsilon - \theta)\, \varpi_{\alpha\beta},$$

$$\begin{aligned}
\cos\rho\sigma = {} & + \left((\alpha_1 + \alpha_2) \sin(\epsilon - \theta) + (\beta_1 + \beta_2) \sin\theta\right) \\
& \qquad \times \left((\eta_1 + \eta_2) \sin(\epsilon - \theta)\, \varpi_{\alpha\xi} + (\xi_1 + \xi_2) \sin\theta \varpi_{\beta\eta}\right) \\
& + \left((\alpha_3 + \alpha_4) \sin(\epsilon - \theta) + (\beta_3 + \beta_4) \sin\theta\right) \\
& \qquad \times \left((\eta_3 + \eta_4) \sin(\epsilon - \theta)\, \varpi_{\alpha\xi} + (\xi_3 + \xi_4) \sin\theta \varpi_{\beta\eta}\right) \\
& + \left((\alpha_5 + \alpha_6) \sin(\epsilon - \theta) + (\beta_5 + \beta_6) \sin\theta\right) \\
& \qquad \times \left((\eta_5 + \eta_6) \sin(\epsilon - \theta)\, \varpi_{\alpha\xi} + (\xi_5 + \xi_6) \sin\theta \varpi_{\beta\eta}\right) \\
= {} & \sin^2(\epsilon - \theta)\, \varpi_{\alpha\xi} \cos(\alpha\eta) + \sin(\epsilon - \theta) \sin\theta\, \varpi_{\beta\eta} \cos(\alpha\xi) \\
& \qquad + \sin(\epsilon - \theta) \sin\theta \varpi_{\alpha\xi} \cos(\beta\eta) + \sin^2\theta \varpi_{\beta\eta} \cos(\beta\xi),
\end{aligned}$$

$$\varpi_{\sigma\lambda} = \sin(\epsilon - \theta)\, \varpi_{\alpha\lambda} + \sin\theta \varpi_{\beta\lambda}.$$

Substituting we have

$$0 = \cos(\lambda\mu)(\sin(\epsilon-\theta)\,\varpi_{\alpha\xi}\,\varpi_{\mu\eta} + \sin\theta\varpi_{\beta\eta}\,\varpi_{\mu\xi})$$
$$(\sin^2(\epsilon-\theta)\,p_\alpha + \sin^2\theta p_\beta + 2\sin\theta\,\sin(\epsilon-\theta)\,\varpi_{\alpha\beta})$$
$$-p\mu(\sin(\epsilon-\theta)\,\varpi_{\alpha\lambda}+\sin\theta\varpi_{\beta\lambda})(\sin^2(\epsilon-\theta)\cos(\alpha\eta)\varpi_{\alpha\xi}+\sin(\epsilon-\theta)\sin\theta\varpi_{\beta\eta}\cos\alpha\xi)$$
$$+\sin(\epsilon-\theta)\,\sin\theta\varpi_{\alpha\xi}\cos(\beta\eta)+\sin^2\theta\varpi_{\beta\eta}\cos(\beta\xi).$$

$$0 = +\sin^3(\epsilon-\theta)\,[\cos(\lambda\mu)\,p_\alpha\,\varpi_{\alpha\xi}\,\varpi_{\mu\eta} - \cos(\alpha\eta)\,p_\mu\,\varpi_{\alpha\lambda}\varpi_{\alpha\xi}]$$
$$+\sin^3\theta\,[\cos(\lambda\mu)\,p_\beta\,\varpi_{\beta\eta}\,\varpi_{\mu\xi} - p_\mu\varpi_{\beta\lambda}\,\varpi_{\beta\eta}\,\cos(\beta\xi)\,]$$
$$+\sin^2\theta\,\sin(\epsilon-\theta)\,[2\cos(\lambda\mu)\,\varpi_{\beta\lambda}\,\varpi_{\mu\xi}\,\varpi_{\alpha\beta} + p_\beta\cos(\lambda\mu)\,\varpi_{\alpha\xi}\,\varpi_{\mu\eta}$$
$$-p_\mu\,\varpi_{\beta\eta}\,\varpi_{\alpha\lambda}\cos(\beta\xi) - p_\mu\,\varpi_{\beta\lambda}\,\varpi_{\beta\eta}\cos(\alpha\xi) - p_\mu\,\varpi_{\beta\lambda}\,\varpi_{\alpha\xi}\cos(\beta\eta)]$$
$$+\sin\theta\,\sin^2(\epsilon-\theta)\,[2\cos(\lambda\mu)\,\varpi_{\alpha\xi}\,\varpi_{\mu\eta}\,\varpi_{\alpha\beta} + \cos(\lambda\mu)\,p_\alpha\,\varpi_{\beta\eta}\,\varpi_{\mu\xi}$$
$$-p_\mu\,\varpi_{\beta\lambda}\cos(\alpha\eta)\,\varpi_{\alpha\xi} - p_\mu\,\varpi_{\alpha\lambda}\,\varpi_{\beta\eta}\,\cos(\alpha\xi) - p_\mu\,\varpi_{\alpha\lambda}\,\varpi_{\alpha\xi}\cos(\beta\eta)].$$

But we have, of course,

$$\frac{\cos\lambda\mu}{p_\mu}\,\varpi_{\mu\eta} = \frac{\cos\alpha\eta}{p_\alpha}\,\varpi_{\alpha\lambda},$$
$$\frac{\cos\lambda\mu}{p_\mu}\,\varpi_{\mu\xi} = \frac{\cos\beta\xi}{p_\beta}\,\varpi_{\beta\lambda}.$$

These conditions make the coefficients of $\sin^3(\epsilon-\theta)$ and $\sin^3\theta$ to vanish. From equating the coefficient of $\sin^2\theta\,\sin(\epsilon-\theta)$ to zero, we have

$$0 = +\varpi_{\beta\eta}\,(2\cos(\lambda\mu)\,\varpi_{\mu\xi}\,\varpi_{\alpha\beta} - p_\mu\,(\varpi_{\beta\lambda}\cos(\alpha\xi) + \varpi_{\alpha\lambda}\cos(\beta\xi)))$$
$$+\varpi_{\alpha\xi}\,(p_\beta\cos(\lambda\mu)\,\varpi_{\mu\eta} - p_\mu\,\varpi_{\beta\lambda}\cos(\beta\eta)),$$
$$0 = +\varpi_{\alpha\xi}\,(2\cos(\lambda\mu)\,\varpi_{\mu\eta}\,\varpi_{\alpha\beta} - p_\mu\,(\varpi_{\beta\lambda}\cos(\alpha\eta) + \varpi_{\alpha\lambda}\cos(\beta\eta)))$$
$$+\varpi_{\beta\eta}\,(p_\alpha\cos(\lambda\mu)\,\varpi_{\mu\xi} - p_\mu\cos(\alpha\xi)\,\varpi_{\alpha\lambda}).$$

We must see if these can be simplified. I write the first thus:

$$0 = -p_\mu\,\varpi_{\beta\lambda}\,(\varpi_{\alpha\xi}\cos(\beta\eta) + \varpi_{\beta\eta}\cos(\alpha\xi)) + 2\cos(\lambda\mu)\,\varpi_{\mu\xi}\,\varpi_{\alpha\beta}\,\varpi_{\beta\eta}$$
$$+\cos(\lambda\mu)\,p_\beta\,\varpi_{\alpha\xi}\,\varpi_{\mu\eta} - \cos(\beta\xi)\,p_\mu\,\varpi_{\alpha\lambda}\,\varpi_{\beta\eta}.$$

Multiply by p_β, substituting from the results already obtained,

$$0 = -p_\mu \varpi_{\beta\lambda}(2\cos(\beta\xi)\varpi_{\alpha\beta}\varpi_{\beta\eta}) + 2\cos\lambda\mu\, p_\beta \varpi_{\mu\xi}\varpi_{\alpha\beta}\varpi_{\beta\eta} + \cos(\lambda\mu)\, p_\beta \varpi_{\alpha\xi}\varpi_{\mu\eta} - \cos(\beta\xi)\, p_\beta p_\mu \varpi_{\alpha\lambda}\varpi_{\beta\eta},$$

$$0 = 2\varpi_{\alpha\beta}\varpi_{\beta\eta}(\cos\lambda\mu\, p_\beta \varpi_{\mu\xi} - p_\mu \cos(\beta\xi)\varpi_{\beta\lambda}) + p_\beta(p_\beta \cos(\lambda\mu)\varpi_{\alpha\xi}\varpi_{\mu\eta} - \cos(\beta\xi)\, p_\mu \varpi_{\alpha\lambda}\varpi_{\beta\eta}).$$

But the first part is zero, whence

$$0 = p_\beta \cos(\lambda\mu)\varpi_{\alpha\xi}\varpi_{\mu\eta} - p_\mu \cos(\beta\xi)\varpi_{\alpha\lambda}\varpi_{\beta\eta};$$

whence this conveys nothing but the known property of chiastic homography (see "Tenth Memoir"),

$$0 = \varpi_{\alpha\xi}\varpi_{\mu\eta}\varpi_{\beta\lambda} - \varpi_{\alpha\lambda}\varpi_{\beta\eta}\varpi_{\mu\xi}.$$

Thus the verification is complete, for the other part reduces in the same way.

III.—On the Homographic Relation between two Cylindroids when one is the Locus of the Impulsive Screws corresponding to Instantaneous Screws on the other.

Let λ, λ' be two screws on a cylindroid whereof α and β are the two principal screws.

Let θ, θ' be the angles which λ and λ' respectively make with α.

Then we have as the coordinates of λ, with reference to six screws of reference, which for convenience we shall take two by two, with equal and opposite pitches to the radii of gyration on three principal axes of the rigid body,

$$\alpha_1\cos\theta + \beta_1\sin\theta, \quad \alpha_2\cos\theta + \beta_2\sin\theta, \ \&c.,$$

and

$$\alpha_1\cos\theta' + \beta_1\sin\theta', \quad \alpha_2\cos\theta' + \beta_2\sin\theta', \ \&c.$$

In like manner, let ρ and ρ' be two screws on a cylindroid, of which the two principal screws are η and ξ.

Let ϕ, ϕ' be the angles which ρ and ρ' make respectively with ξ.

Then the coordinates of ρ are

$$\eta_1 \cos\phi + \xi_1 \sin\phi, \quad \eta_2 \cos\phi + \xi_2 \sin\phi,$$

and of ρ',

$$\eta_1 \cos\phi' + \xi_1 \sin\phi', \quad \eta_2 \cos\phi' + \xi_2 \sin\phi', \quad \&c.$$

We shall now suppose that the two cylindroids α, β and η, ξ are so circumstanced that the latter is the locus of the impulsive wrenches corresponding to the several instantaneous screws on the former with respect to a certain rigid body. We shall further assume that ρ is the impulsive screw which has λ as its instantaneous screw, and that the relation of ρ' to λ' is of the same nature.

If, however, the four screws λ, λ', ρ, ρ' possess the relations thus indicated, it is necessary that they satisfy the conditions already proved. These are two-fold, and they are expressed by the following equations, as already shown:

$$\frac{p_\lambda}{\cos\lambda\rho} \cos\lambda'\rho + \frac{p_{\lambda'}}{\cos\lambda'\rho'} \cos\lambda\rho' = 2\varpi_{\lambda\lambda'},$$

$$\frac{p_\lambda}{\cos\lambda\rho} \varpi_{\lambda'\rho} = \frac{p_{\lambda'}}{\cos\lambda'\rho'} \varpi_{\lambda\rho'}.$$

We shall arbitrarily choose λ' and ρ', so as to satisfy the conditions

$$\varpi_{\lambda'\rho} = 0, \quad \varpi_{\lambda\rho'} = 0,$$

and thus the second of the two equations is satisfied. These two equations will give θ' as a function of ϕ, and ϕ' as a function of θ. We can thus eliminate θ' and ϕ' from the first of the two equations, and the result will be a relation connecting θ and ϕ. This equation I propose to investigate. It will exhibit the relation between any instantaneous screw θ on one cylindroid, and the corresponding impulsive screw ϕ on the other.

It will be observed that when the two cylindroids are given, the required equation is completely defined. The homographic relations of ρ and λ is thus completely determined by the geometrical relations of the two cylindroids. The investigation is one of considerable interest in the theory, and it may serve as an illustration of the use of screw coordinates.

Since $\varpi_{\lambda\rho'} = 0$, we have

$$\Sigma p_1(\alpha_1 \cos\theta + \beta_1 \sin\theta)(\eta_1 \cos\phi' + \xi_1 \sin\phi') = 0,$$

or

$$0 = \varpi_{\alpha\eta} \cos\theta \cos\phi' + \varpi_{\beta\xi} \sin\theta \sin\phi' + \varpi_{\beta\eta} \sin\theta \cos\phi' + \varpi_{\alpha\xi} \cos\theta \sin\phi';$$

and, since $\varpi_{\lambda'\rho} = 0$,

$$0 = \varpi_{\alpha\eta} \cos\theta' \cos\phi + \varpi_{\beta\xi} \sin\theta' \sin\phi + \varpi_{\beta\eta} \sin\theta' \cos\phi + \varpi_{\alpha\xi} \cos\theta' \sin\phi.$$

As the equations from which θ' and ϕ' are to be eliminated are homogeneous functions of the sines and cosines of these angles, we may put

$$\sin\theta' = -\varpi_{\alpha\eta} \cos\phi - \varpi_{\alpha\xi} \sin\phi,$$
$$\cos\theta' = +\varpi_{\beta\eta} \cos\phi + \varpi_{\beta\xi} \sin\phi,$$
$$\sin\phi' = +\varpi_{\alpha\eta} \cos\theta + \varpi_{\beta\eta} \sin\theta,$$
$$\cos\phi' = -\varpi_{\beta\xi} \sin\theta - \varpi_{\alpha\xi} \cos\theta.$$

We have the following general formula for the cosine of the angle between two screws $\lambda\rho$:

$$\cos(\lambda\rho) = (\lambda_1 + \lambda_2)(\rho_1 + \rho_2) + (\lambda_3 + \lambda_4)(\rho_3 + \rho_4) + (\lambda_5 + \lambda_6)(\rho_5 + \rho_6);$$

whence, substituting for λ_1, λ_2, ρ_1, ρ_2, &c., we obtain

$$\begin{aligned}\cos(\lambda\rho) = {} & (\alpha_1 \cos\theta + \beta_1 \sin\theta + \alpha_2 \cos\theta + \beta_2 \sin\theta) \\ & \times (\eta_1 \cos\phi + \xi_1 \sin\phi + \eta_2 \cos\phi + \xi_2 \sin\phi) \\ & + (\alpha_3 \cos\theta + \beta_3 \sin\theta + \alpha_4 \cos\theta + \beta_4 \sin\theta) \\ & \times (\eta_3 \cos\phi + \xi_3 \sin\phi + \eta_4 \cos\phi + \xi_4 \sin\phi) \\ & + (\alpha_5 \cos\theta + \beta_5 \sin\theta + \alpha_6 \cos\theta + \beta_6 \sin\theta) \\ & \times (\eta_5 \cos\phi + \xi_5 \sin\phi + \eta_6 \cos\phi + \xi_6 \sin\phi),\end{aligned}$$

or

$$\begin{aligned}\cos\lambda\rho &= \cos\theta \cos\phi \cos(\alpha\eta) + \cos\theta \sin\phi \cos(\alpha\xi) \\ &\quad + \sin\theta \cos\phi \cos(\beta\eta) + \sin\theta \sin\phi \cos(\beta\xi), \\ \cos\lambda\rho' &= \cos\theta \cos\phi' \cos(\alpha\eta) + \cos\theta \sin\phi' \cos(\alpha\xi) \\ &\quad + \sin\theta \cos\phi' \cos(\beta\eta) + \sin\theta \sin\phi' \cos(\beta\xi), \\ \cos\lambda'\rho &= \cos\theta' \cos\phi \cos(\alpha\eta) + \cos\theta' \sin\phi \cos(\alpha\xi) \\ &\quad + \sin\theta' \cos\phi \cos(\beta\eta) + \sin\theta' \sin\phi \cos(\beta\xi), \\ \cos\lambda'\rho' &= \cos\theta' \cos\phi' \cos(\alpha\eta) + \cos\theta' \sin\phi' \cos(\alpha\xi) \\ &\quad + \sin\theta' \cos\phi' \cos(\beta\eta) + \sin\theta' \sin\phi' \cos(\beta\xi).\end{aligned}$$

We have also

$$p_\lambda = \cos^2\theta p_\alpha + \sin^2\theta p_\beta,$$

$$p_{\lambda'} = \cos^2\theta' p_\alpha + \sin^2\theta' p_\beta,$$

$$\varpi_{\lambda\lambda'} = \Sigma p_1 (\alpha_1 \cos\theta + \beta_1 \sin\theta)(\alpha_1 \cos\theta' + \beta_1 \sin\theta')$$

$$= \cos\theta \cos\theta' \varpi_\alpha + \sin\theta \sin\theta' p_\beta.$$

Substitute in the equation

$$0 = p_\lambda \cos\lambda'\rho \cos\lambda'\rho' + p_{\lambda'} \cos\lambda\rho' \cos\lambda\rho - 2\varpi_{\lambda\lambda'} \cos\lambda\rho \cos\lambda'\rho',$$

and we obtain

$$\begin{aligned}
0 = {} & (\cos^2\theta p_\alpha + \sin^2\theta p_\beta) \\
& \times (\cos\theta' \cos\phi \cos(\alpha\eta) + \cos\theta' \sin\phi \cos(\alpha\xi) + \sin\theta' \cos\phi \cos(\beta\eta) \\
& \qquad + \sin\theta' \sin\phi \cos(\beta\xi)) \\
& \times (\cos\theta' \cos\phi' \cos(\alpha\eta) + \cos\theta' \sin\phi' \cos(\alpha\xi) + \sin\theta' \cos\phi' \cos(\beta\eta) \\
& \qquad + \sin\theta' \sin\phi' \cos(\beta\xi)) + (\cos^2\theta' p_\alpha + \sin^2\theta' p_\beta) \\
& \times (\cos\theta \cos\phi' \cos(\alpha\eta) + \cos\theta \sin\phi' \cos(\alpha\xi) + \sin\theta \cos\phi' \cos(\beta\eta) \\
& \qquad + \sin\theta \sin\phi' \cos(\beta\xi)) \\
& \times (\cos\theta \cos\phi \cos(\alpha\eta) + \cos\theta \sin\phi \cos(\alpha\xi) + \sin\theta \cos\phi \cos(\beta\eta) \\
& \qquad + \sin\theta \sin\phi \cos(\beta\xi)) - 2(\cos\theta \cos\theta' p_\alpha + \sin\theta \sin\theta' p_\beta) \\
& \times (\cos\theta \cos\phi \cos(\alpha\eta) + \cos\theta \sin\phi \cos(\alpha\xi) + \sin\theta \cos\phi \cos(\beta\eta) \\
& \qquad + \sin\theta \sin\phi \cos(\beta\xi)) \\
& \times (\cos\theta' \cos\phi' \cos(\alpha\eta) + \cos\theta' \sin\phi' \cos(\alpha\xi) + \sin\theta' \cos\phi' \cos(\beta\eta) \\
& \qquad + \sin\theta' \sin\phi' \cos(\beta\xi)).
\end{aligned}$$

To reduce this equation, we first collect the coefficient of p_α, which consists of 48 terms which reduce as follows:—

$$\begin{aligned}
& + \cos^2\theta \cos^2\theta' (\cos\phi \cos(\alpha\eta) + \sin\phi \cos(\alpha\xi)) \\
& \qquad \times (\cos\phi' \cos(\alpha\eta) + \sin\phi' \cos(\alpha\xi)) \\
& + \cos^2\theta \sin^2\theta' (\cos\phi \cos(\beta\eta) + \sin\phi \cos(\beta\xi)) \\
& \qquad \times (\cos\phi' \cos(\beta\eta) + \sin\phi' \cos(\beta\xi)) \\
& + \cos^2\theta \sin\theta' \cos\theta' (\cos\phi \cos(\alpha\eta) + \sin\phi \cos(\alpha\xi)) \\
& \qquad \times (\cos\phi' \cos(\beta\eta) + \sin\phi' \cos(\beta\xi)) \\
& + (\cos\phi' \cos(\alpha\eta) + \sin\phi' \cos(\alpha\xi))(\cos\phi \cos(\beta\eta) + \sin\phi \cos(\beta\xi))
\end{aligned}$$

$$
\begin{aligned}
&+\cos^2\theta\cos^2\theta'(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))\\
&\qquad\times(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))\\
&+\cos^2\theta'\sin^2\theta(\cos\phi\cos(\beta\eta)+\sin\phi\cos(\beta\xi))\\
&\qquad\times(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))\\
&+\cos^2\theta'\sin\theta\cos\theta(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))\\
&\qquad\times(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))\\
&\quad+(\cos\phi\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))\\
&-2\cos^2\theta\cos^2\theta'(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))\\
&\qquad\times(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))\\
&-2\sin\theta\cos\theta\sin\theta'\cos\theta'(\cos\phi\cos(\beta\eta)+\sin\phi\cos(\beta\xi))\\
&\qquad\times(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))\\
&-2\cos^2\theta'\sin\theta\cos\theta(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))\\
&\qquad\times(\cos\phi\cos(\beta\eta)+\sin\phi\cos(\beta\xi))\\
&-2\cos^2\theta\sin\theta'\cos\theta'(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))\\
&\qquad\times(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))\\
&=\sin(\theta-\theta')^2(\cos\phi\cos(\beta\eta)+\sin\phi\cos(\beta\xi))\\
&\qquad\times(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))\\
&+\cos^2\theta\sin\theta'\cos\theta'(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))\\
&\qquad\times(\cos\phi\cos(\beta\eta)+\sin\phi\cos(\beta\xi))\\
&-(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))\\
&+\cos^2\theta'\sin\theta\cos\theta\left[\begin{array}{l}(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))\\\qquad\times(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))\\-(\cos\phi\cos(\beta\eta)+\sin\phi\cos(\beta\xi))\\\qquad\times(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))\end{array}\right]\\
&=\sin^2(\theta-\theta')(\cos\phi\cos(\beta\eta)+\sin\phi\cos(\beta\xi))\\
&\qquad\times(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))\\
&+\cos\theta\cos\theta'\sin(\theta-\theta')\sin(\phi-\phi')(\cos(\alpha\xi)\cos(\beta\eta)\\
&\qquad\qquad-\cos(\alpha\eta)\cos(\beta\xi)).
\end{aligned}
$$

In like manner we compute the coefficient of p_β:

$$+ \sin^2\theta\cos^2\theta'(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))$$
$$+ \sin^2\theta\sin^2\theta'(\cos\phi\cos(\beta\eta)+\sin\phi\cos(\beta\xi))(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))$$
$$+ \sin^2\theta'\cos^2\theta(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))$$
$$+ \sin^2\theta'\sin^2\theta(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))(\cos\phi\cos(\beta\eta)+\sin\phi\cos(\beta\xi))$$
$$+ \sin^2\theta\sin\theta'\cos\theta'\left[\begin{array}{l}(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))\\ \qquad\times(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))\\ -(\cos\phi\cos(\beta\eta)+\sin\phi\cos(\beta\xi))\\ \qquad\times(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))\end{array}\right]$$
$$+ \sin^2\theta'\sin\theta\cos\theta\left[\begin{array}{l}(\cos\phi\cos(\beta\eta)+\sin\phi\cos(\beta\xi))\\ \qquad\times(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))\\ -(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))\\ \qquad\times(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))\end{array}\right]$$
$$- 2\sin^2\theta\sin^2\theta'(\cos\phi\cos(\beta\eta)+\sin\phi\cos(\beta\xi))\times(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))$$
$$- 2\sin\theta\cos\theta\sin\theta'\cos\theta'(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))\times(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))$$
$$- 2\sin^2\theta'\sin\theta\cos\theta(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))\times(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))$$
$$- 2\sin^2\theta\sin\theta'\cos\theta'(\cos\phi\cos(\beta\eta)+\sin\phi\cos(\beta\xi))\times(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))$$
$$= \sin^2(\theta-\theta')(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))\times(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))$$
$$+ \sin\theta\sin\theta'\sin(\theta-\theta')\left[\begin{array}{l}(\cos\phi\cos\alpha\eta+\sin\phi\cos\alpha\xi)\\ \qquad\times(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))\\ -(\cos\phi\cos\beta\eta+\sin\phi\cos\beta\xi)\\ \qquad\times(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))\end{array}\right]$$
$$= \sin^2(\theta-\theta')(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))\times(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))$$
$$+ \sin\theta\sin\theta'\sin(\theta-\theta')\sin(\phi-\phi')(\cos\alpha\xi\cos\beta\eta-\cos\beta\xi\cos\alpha\eta).$$

The equation thus reduces to

$$\begin{aligned}
&p_\alpha \sin^2(\theta-\theta')(\cos\phi\cos(\beta\eta)+\sin\phi\cos(\beta\xi))\\
&\qquad\times(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))\\
&+p_\alpha\cos\theta\cos\theta'\sin(\theta-\theta')\sin(\phi-\phi')\\
&\qquad\times(\cos(\alpha\xi)\cos(\beta\eta)-\cos(\alpha\eta)\cos(\beta\xi))\\
&+p_\beta\sin^2(\theta-\theta')(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))\\
&\qquad\times(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))\\
&+p_\beta\sin\theta\sin\theta'\sin(\theta-\theta')\sin(\phi-\phi')\\
&\qquad\times(\cos(\alpha\xi)\cos(\beta\eta)-\cos(\alpha\eta)\cos(\beta\xi))=0.
\end{aligned}$$

This can be further simplified as follows, by the elimination of θ' and ϕ'.

We may at once reject the factor $\sin(\theta-\theta')$, and the equation becomes

$$\begin{aligned}
&\sin(\theta-\theta')\Big[+p_\alpha(\cos\phi\cos(\beta\eta)+\sin\phi\cos(\beta\xi))\\
&\qquad\times(\cos\phi'\cos(\beta\eta)+\sin\phi'\cos(\beta\xi))\\
&\quad+p_\beta(\cos\phi\cos(\alpha\eta)+\sin\phi\cos(\alpha\xi))\\
&\qquad\times(\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos(\alpha\xi))\Big]\\
&+\sin(\phi-\phi')(\cos\alpha\xi\cos\beta\eta-\cos\alpha\eta\cos\beta\xi)\\
&\qquad\times(p_\alpha\cos\theta\cos\theta'+p_\beta\sin\theta\sin\theta')=0.
\end{aligned}$$

Substituting, from the expressions already given,

$$\sin(\theta-\theta')=\sin\theta(\varpi_{\beta\eta}\cos\phi+\varpi_{\beta\xi}\sin\phi)+\cos\theta(\varpi_{\alpha\eta}\cos\phi+\varpi_{\alpha\xi}\sin\phi),$$

$$\sin(\phi-\phi')=-\sin\phi(\varpi_{\beta\xi}\sin\theta+\varpi_{\alpha\xi}\cos\theta)-\cos\phi(\varpi_{\alpha\eta}\cos\theta+\varpi_{\beta\eta}\sin\theta).$$

$$\begin{aligned}
&\cos\phi'\cos\beta\eta+\sin\phi'\cos\beta\xi\\
&\quad=(-\varpi_{\beta\xi}\sin\theta-\varpi_{\alpha\xi}\cos\theta)\cos\beta\eta+(\varpi_{\alpha\eta}\cos\theta+\varpi_{\beta\eta}\sin\theta)\cos\beta\xi\\
&\quad=\cos\theta(\varpi_{\alpha\eta}\cos\beta\xi-\varpi_{\alpha\xi}\cos\beta\eta)+\sin\theta(\varpi_{\beta\eta}\cos\beta\xi-\varpi_{\beta\xi}\cos\beta\eta).
\end{aligned}$$

$$\begin{aligned}
&\cos\phi'\cos(\alpha\eta)+\sin\phi'\cos\alpha\xi\\
&\quad=\cos\alpha\eta(-\varpi_{\beta\xi}\sin\theta-\varpi_{\alpha\xi}\cos\theta)+(\varpi_{\alpha\eta}\cos\theta+\varpi_{\beta\eta}\sin\theta)\cos\alpha\xi\\
&\quad=\cos\theta(\varpi_{\alpha\eta}\cos\alpha\xi-\cos\alpha\eta\,\varpi_{\alpha\xi})+\sin\theta(\varpi_{\beta\eta}\cos\alpha\xi-\varpi_{\beta\xi}\cos\alpha\eta).
\end{aligned}$$

$$\begin{aligned}
&p_\alpha\cos\theta\cos\theta'+p_\beta\sin\theta\sin\theta'\\
&\quad=p_\alpha\cos\theta(\varpi_{\beta\eta}\cos\phi+\varpi_{\beta\xi}\sin\phi)+p_\beta\sin\theta(-\varpi_{\alpha\eta}\cos\phi-\varpi_{\alpha\xi}\sin\phi)\\
&\quad=p_\alpha\varpi_{\beta\eta}\cos\theta\cos\phi+p_\alpha\varpi_{\beta\xi}\cos\theta\sin\phi-p_\beta\varpi_{\alpha\eta}\sin\theta\cos\phi\\
&\qquad\qquad-p_\beta\varpi_{\alpha\xi}\sin\theta\sin\phi.
\end{aligned}$$

Thus the equation becomes

$$\begin{aligned}0 = {} & p_\alpha(\cos\phi\cos\beta\eta + \sin\phi\cos\beta\xi)(\cos\theta(\varpi_{\alpha\eta}\cos\beta\xi - \varpi_{\alpha\xi}\cos\beta\eta) \\ & \qquad + \sin\theta(\varpi_{\beta\eta}\cos\beta\xi - \varpi_{\beta\xi}\cos\beta\eta)) \\ & + p_\beta(\cos\phi\cos\alpha\eta + \sin\phi\cos\alpha\xi)(\cos\theta(\varpi_{\alpha\eta}\cos\alpha\xi - \cos\alpha\eta\,\varpi_{\alpha\xi}) \\ & \qquad + \sin\theta(\varpi_{\beta\eta}\cos\alpha\xi - \varpi_{\beta\xi}\cos\alpha\eta)) \\ & - (\cos\alpha\xi\cos\beta\eta - \cos\alpha\eta\cos\beta\xi)(p_\alpha\varpi_{\beta\eta}\cos\theta\cos\phi + p_\alpha\varpi_{\beta\xi}\cos\theta\sin\phi \\ & \qquad - p_\beta\varpi_{\alpha\eta}\sin\theta\cos\phi - p_\beta\varpi_{\alpha\xi}\sin\theta\sin\phi),\end{aligned}$$

whence finally the required homographic relation defining θ and ϕ.

$$\begin{aligned}0 = {} & \cos\theta\cos\phi \left[\begin{array}{l} + p_\alpha\cos\beta\eta\,(\varpi_{\alpha\eta}\cos\beta\xi - \varpi_{\alpha\xi}\cos\beta\eta) \\ + p_\beta\cos\alpha\eta\,(\varpi_{\alpha\eta}\cos\alpha\xi - \varpi_{\alpha\xi}\cos\alpha\eta) \\ - p_\alpha\varpi_{\beta\eta}(\cos\alpha\xi\cos\beta\eta - \cos\alpha\eta\cos\beta\xi)\end{array}\right] \\ & + \cos\theta\sin\phi \left[\begin{array}{l} + p_\alpha\cos\beta\xi\,(\varpi_{\alpha\eta}\cos\beta\xi - \varpi_{\alpha\xi}\cos\beta\eta) \\ + p_\beta\cos\alpha\xi\,(\varpi_{\alpha\eta}\cos\alpha\xi - \cos\alpha\eta\,\varpi_{\alpha\xi}) \\ - p_\alpha\varpi_{\beta\xi}(\cos\alpha\xi\cos\beta\eta - \cos\alpha\eta\cos\beta\xi)\end{array}\right] \\ & + \sin\theta\cos\phi \left[\begin{array}{l} + p_\beta\cos\alpha\eta\,(\varpi_{\beta\eta}\cos\alpha\xi - \varpi_{\beta\xi}\cos\alpha\eta) \\ + p_\alpha\cos\beta\eta\,(\varpi_{\beta\eta}\cos\beta\xi - \varpi_{\beta\xi}\cos\beta\eta) \\ + p_\beta\varpi_{\alpha\eta}(\cos\alpha\xi\cos\beta\eta - \cos\alpha\eta\cos\beta\xi)\end{array}\right] \\ & + \sin\theta\sin\phi \left[\begin{array}{l} + p_\beta\cos\alpha\xi\,(\varpi_{\beta\eta}\cos\alpha\xi - \varpi_{\beta\xi}\cos\alpha\eta) \\ + p_\alpha\cos\beta\xi\,(\varpi_{\beta\eta}\cos\beta\xi - \varpi_{\beta\xi}\cos\beta\eta) \\ + p_\beta\varpi_{\alpha\xi}(\cos\alpha\xi\cos\beta\eta - \cos\alpha\eta\cos\beta\xi)\end{array}\right]\end{aligned}$$

The following is a method of testing this formula. We may write it as follows:

$$\begin{aligned} & + \varpi_{\alpha\eta}\left[\begin{array}{l}(p_\alpha\cos\beta\eta\cos\beta\xi + p_\beta\cos\alpha\xi\cos\alpha\eta)\cos\theta\cos\phi \\ (p_\alpha\cos\beta\xi\cos\beta\xi + p_\beta\cos^2\alpha\xi)\cos\theta\sin\phi \\ (p_\beta\cos\alpha\xi\cos\beta\eta - p_\beta\cos\alpha\eta\cos\beta\xi)\sin\theta\cos\phi\end{array}\right] \\ & + \varpi_{\alpha\xi}\left[\begin{array}{l}(-p_\alpha\cos^2\beta\eta - p_\beta\cos^2\alpha\eta)\cos\theta\cos\phi \\ (-p_\alpha\cos\beta\xi\cos\beta\eta - p_\beta\cos\alpha\eta\cos\alpha\xi)\cos\theta\sin\phi \\ (+p_\beta\cos\alpha\xi\cos\beta\eta - p_\beta\cos\alpha\eta\cos\beta\xi)\sin\theta\sin\phi\end{array}\right]\end{aligned}$$

$$
\begin{aligned}
+\varpi_{\beta\eta}\quad & (-p_\alpha\cos\alpha\xi\cos\beta\eta + p_\alpha\cos\alpha\eta\cos\beta\xi)\cos\theta\cos\phi\\
& (+p_\beta\cos\alpha\eta\cos\alpha\xi + p_\alpha\cos\beta\eta\cos\beta\xi)\sin\theta\cos\phi\\
& (+p_\beta\cos^2\alpha\xi \qquad\quad + p_\alpha\cos^2\beta\xi \qquad\quad)\sin\theta\sin\phi
\end{aligned}
$$

$$
\begin{aligned}
+\varpi_{\beta\xi}\quad & (-p_\alpha\cos\alpha\xi\cos\beta\eta + p_\alpha\cos\alpha\eta\cos\beta\xi)\cos\theta\sin\phi\\
& (-p_\beta\cos^2\alpha\eta \qquad\quad - p_\alpha\cos^2\beta\eta \qquad\quad)\sin\theta\cos\phi\\
& (-p_\alpha\cos\beta\xi\cos\beta\eta - p_\beta\cos\alpha\eta\cos\alpha\xi)\sin\theta\sin\phi.
\end{aligned}
$$

Let us now assume that θ is one of the screws of zero pitch, so that

$$p_\alpha\cos^2\theta + p_\beta\sin^2\theta = 0.$$

We shall then make

$$p_\alpha = H\sin^2\theta,$$
$$p_\beta = -H\cos^2\theta,$$

and, by substitution, the expression becomes :

$$
\begin{aligned}
+\varpi_{\alpha\eta}H\Big[\quad & +\sin^2\theta\cos\theta\cos\phi\cos\beta\eta\cos\beta\xi - \cos^3\theta\cos\phi\cos\alpha\xi\cos\alpha\eta\\
& +\sin^2\theta\cos\theta\sin\phi\cos\beta\xi\cos\beta\xi - \cos^3\theta\sin\phi\cos^2\alpha\xi\\
& -\sin\theta\cos^2\theta\cos\alpha\xi\cos\beta\eta\cos\phi + \sin\theta\cos^2\theta\cos\alpha\eta\cos\beta\xi\cos\phi\ \Big]
\end{aligned}
$$

$$
\begin{aligned}
+\varpi_{\alpha\xi}H\Big[\quad & -\sin^2\theta\cos^2\beta\eta\cos\theta\cos\phi \qquad + \cos^2\theta\cos^2\alpha\eta\cos\theta\cos\phi\\
& -\sin^2\theta\cos\theta\sin\phi\cos\beta\xi\cos\beta\eta + \cos^3\theta\sin\phi\cos\alpha\eta\cos\alpha\xi\\
& -\sin\theta\cos^2\theta\sin\phi\cos\alpha\xi\cos\beta\eta + \sin\theta\cos^2\theta\sin\phi\cos\alpha\eta\cos\beta\xi\ \Big]
\end{aligned}
$$

$$
\begin{aligned}
+\varpi_{\beta\eta}H\Big[\quad & -\sin^2\theta\cos\theta\cos\phi\cos\alpha\xi\cos\beta\eta + \sin^2\theta\cos\theta\cos\phi\cos\alpha\eta\cos\beta\xi\\
& -\sin\theta\cos^2\theta\cos\phi\cos\alpha\eta\cos\alpha\xi + \sin^3\theta\cos\phi\cos\beta\eta\cos\beta\xi\\
& -\sin\theta\cos^2\theta\sin\theta\cos^2\alpha\xi \qquad + \sin^3\theta\sin\phi\cos^2\beta\xi\ \Big]
\end{aligned}
$$

$$
\begin{aligned}
+\varpi_{\beta\xi}H\Big[\quad & -\sin^2\theta\cos\theta\sin\phi\cos\alpha\xi\cos\beta\eta + \sin^2\theta\cos\theta\sin\phi\cos\alpha\eta\cos\beta\xi\\
& +\sin\theta\cos^2\theta\cos\phi\cos^2\alpha\eta \qquad - \sin^3\theta\cos\phi\cos^2\beta\eta\\
& -\sin^3\theta\sin\phi\cos\beta\xi\cos\beta\eta \qquad + \sin\theta\cos^2\theta\sin\phi\cos\alpha\eta\cos\alpha\xi\ \Big]
\end{aligned}
$$

In this case the impulsive wrench would be perpendicular to the instantaneous screw, so that

$$\cos(\lambda\rho) = 0;$$

or

$$
\begin{aligned}
0 = {} & \cos\theta\cos\phi\cos\alpha\eta + \cos\theta\sin\phi\cos\alpha\xi + \sin\theta\cos\phi\cos\beta\eta\\
& + \sin\theta\sin\phi\cos\beta\xi.
\end{aligned}
$$

If we multiply this equation by $H\cos^2\theta\cos\alpha\xi$, and add it to the coefficient of $\varpi_{\alpha\eta}$, we obtain

$$+\varpi_{\alpha\eta}H\ \sin\theta\cos\theta\cos\beta\xi\begin{bmatrix}+\sin\theta\cos\phi\cos\beta\eta+\sin\theta\sin\phi\cos\beta\xi\\+\cos\theta\cos\psi\cos\alpha\eta+\cos\theta\sin\phi\cos\alpha\xi\end{bmatrix}$$

which vanishes, as it clearly ought to do.

If we multiply this equation by $H\cos^2\theta\cos\alpha\eta$, and add it to the coefficient of $\varpi_{\alpha\xi}$, we get

$$-\varpi_{\alpha\xi}H\ \sin\theta\cos\theta\cos\beta\eta\begin{bmatrix}+\sin\theta\cos\phi\cos\beta\eta+\sin\theta\sin\phi\cos\beta\xi\\+\cos\theta\cos\phi\cos\alpha\eta+\cos\theta\sin\phi\cos\alpha\xi\end{bmatrix}$$

In the same way, if we multiply the equation by $-H\sin^2\theta\cos\beta\xi$, we have for $\varpi_{\beta\eta}$

$$-\varpi_{\beta\eta}H\ \sin\theta\cos\theta\cos\alpha\xi\begin{bmatrix}+\sin\theta\cos\phi\cos\beta\eta+\sin\theta\sin\phi\cos\beta\xi\\+\cos\theta\cos\phi\cos\alpha\eta+\cos\theta\sin\phi\cos\alpha\xi\end{bmatrix}$$

Finally, if we multiply the equation by $H\sin^2\theta\cos\beta\eta$, and add it to the coefficient of $\varpi_{\beta\eta}$, we get

$$+\varpi_{\beta\xi}H\ \sin\theta\cos\theta\cos\alpha\eta\begin{bmatrix}+\sin\theta\cos\phi\cos\beta\eta+\sin\theta\sin\phi\cos\beta\xi\\+\cos\theta\cos\phi\cos\alpha\eta+\cos\theta\sin\phi\cos\alpha\xi\end{bmatrix}$$

whence we see, that the homographic equations is satisfied as it should be. This provides a complete verification of the work.

IV.—Another Investigation of the Homographic Equation.

The following is another method of finding the homographic equation already obtained. It must be of the form

$$0 = A\sin\theta\sin\phi + B\sin\theta\cos\phi + C\cos\theta\sin\phi + D\cos\theta\cos\phi.$$

We have the following particular cases:

(1). If θ be either of the two screws of zero pitch, then ϕ must be at right angles to it, so that if

$$p_\alpha\cos^2\theta + p_\beta\sin^2\theta = 0;$$

then will

$$0 = \cos\theta\cos\phi\cos\alpha\eta + \cos\theta\sin\phi\cos\alpha\xi + \sin\theta\cos\phi\cos\beta\eta + \sin\theta\sin\phi\cos\beta\xi.$$

From the first equation

$$\tan\phi = -\frac{B\sin\theta + D\cos\theta}{A\sin\theta + C\cos\theta}$$

$$= -\frac{B\sin\theta + D\cos\theta}{A\sin\theta + C\cos\theta},$$

$$\tan\phi = -\frac{\cos\theta\cos\alpha\eta + \sin\theta\cos\beta\eta}{\cos\theta\cos\alpha\xi + \sin\theta\cos\beta\xi},$$

$$\frac{B\sin\theta + D\cos\theta}{A\sin\theta + C\cos\theta} = \frac{\sin\theta\cos\beta\eta + \cos\theta\cos\alpha\eta}{\sin\theta\cos\beta\xi + \cos\theta\cos\alpha\xi};$$

or multiplying out

$$+\sin^2\theta\,(B\cos\beta\xi - A\cos\beta\eta)$$
$$+\cos^2\theta\,(D\cos\alpha\xi - C\cos\alpha\eta)$$
$$+\sin\theta\cos\theta\,(B\cos\alpha\xi + D\cos\beta\xi - A\cos\alpha\eta - C\cos\beta\eta) = 0.$$

But we have, to within a factor, $\sin\theta = \pm\sqrt{-p_\alpha}$, $\cos\theta = \sqrt{+p_\beta}$, whence the conditions are

$$0 = p_\alpha(B\cos\beta\xi - A\cos\beta\eta) - p_\beta(D\cos\alpha\xi - C\cos\alpha\eta),$$
$$0 = B\cos\alpha\xi + D\cos\beta\xi - A\cos\alpha\eta - C\cos\beta\eta.$$

We require one more relation between A, B, C, D, in order to determine the homographic relation. It can be thus obtained.

Let λ be one of the screws of zero pitch on the instantaneous cylindroid, and let ρ be the corresponding screw on the impulsive cylindroid.

Let ρ' be a screw reciprocal to λ, and λ' reciprocal to ρ. Then will ρ' and λ' also fulfil the homographic relation. Let θ, θ' be the angles corresponding to λ, λ', and let ϕ, ϕ' correspond to ρ, ρ'. Then we have

$$p_\alpha\cos^2\theta + p_\beta\sin^2\theta = 0;$$

we have also

$$0 = \cos\theta\cos\phi\cos\alpha\eta + \cos\theta\sin\phi\cos\alpha\xi + \sin\theta\cos\phi\cos\beta\eta + \sin\theta\sin\phi\cos\beta\xi,$$

and

$$0 = \varpi_{\alpha\eta}\cos\theta\cos\phi' + \varpi_{\beta\xi}\sin\theta\sin\phi' + \varpi_{\beta\eta}\sin\theta\cos\phi' + \varpi_{\alpha\xi}\cos\theta\sin\phi',$$
$$0 = \varpi_{\alpha\eta}\cos\theta'\cos\phi + \varpi_{\beta\xi}\sin\theta'\sin\phi + \varpi_{\beta\eta}\sin\theta'\cos\phi + \varpi_{\alpha\xi}\cos\theta'\sin\phi.$$

From these two equations we can determine $\tan\theta'$ and $\tan\phi'$ as follows:

$$\tan\phi' = -\frac{\varpi_{\beta\eta}\tan\theta + \varpi_{\alpha\eta}}{\varpi_{\beta\xi}\tan\theta + \varpi_{\alpha\xi}},$$

$$\tan\theta' = -\frac{\varpi_{\alpha\xi}\tan\phi + \varpi_{\alpha\eta}}{\varpi_{\beta\xi}\tan\phi + \varpi_{\beta\eta}};$$

$$\tan\phi = -\frac{\tan\theta\cos\beta\eta + \cos\alpha\eta}{\tan\theta\cos\beta\xi + \cos\alpha\xi},$$

$$\tan\theta' = \frac{-\varpi_{\alpha\xi}(-\tan\theta\cos\beta\eta - \cos\alpha\eta) - \varpi_{\alpha\eta}(\tan\theta\cos\beta\xi + \cos\alpha\xi)}{\varpi_{\beta\xi}(-\tan\theta\cos\beta\eta - \cos\alpha\eta) + \varpi_{\beta\eta}(\tan\theta\cos\beta\xi + \cos\alpha\xi)}$$

$$= \frac{\tan\theta(\varpi_{\alpha\xi}\cos\beta\eta - \varpi_{\beta\eta}\cos\beta\xi) + \varpi_{\alpha\xi}\cos\alpha\eta - \varpi_{\alpha\eta}\cos\alpha\xi}{\tan\theta(\varpi_{\beta\eta}\cos\beta\xi - \varpi_{\beta\xi}\cos\beta\eta) + \varpi_{\beta\eta}\cos\alpha\xi - \varpi_{\beta\xi}\cos\alpha\eta}.$$

Substitute in

$$A\tan\theta'\tan\phi' + B\tan\theta' + C\tan\phi' + D = 0.$$

and we get

$$\begin{aligned}
0 = {} & A(-\varpi_{\beta\eta}\tan\theta - \varpi_{\alpha\eta})(\tan\theta(\varpi_{\alpha\xi}\cos\beta\eta - \varpi_{\alpha\eta}\cos\beta\xi) \\
& \qquad + \varpi_{\alpha\xi}\cos\alpha\eta - \varpi_{\alpha\eta}\cos\alpha\xi) \\
& + B(\tan\theta(\varpi_{\alpha\xi}\cos\beta\eta - \varpi_{\alpha\eta}\cos\beta\xi) + \varpi_{\alpha\xi}\cos\alpha\eta - \varpi_{\alpha\eta}\cos\alpha\xi) \\
& \qquad \times(\varpi_{\beta\xi}\tan\theta + \varpi_{\alpha\xi}) \\
& + C(-\varpi_{\beta\eta}\tan\theta - \varpi_{\alpha\eta})(\tan\theta(\varpi_{\beta\eta}\cos\beta\xi - \varpi_{\beta\xi}\cos\beta\eta) \\
& \qquad + \varpi_{\beta\eta}\cos\alpha\xi - \varpi_{\beta\xi}\cos\alpha\eta) \\
& + D(\varpi_{\beta\xi}\tan\theta + \varpi_{\alpha\xi})(\tan\theta(\varpi_{\beta\eta}\cos\beta\xi - \varpi_{\beta\xi}\cos\beta\eta) \\
& \qquad + \varpi_{\beta\eta}\cos\alpha\xi - \varpi_{\beta\xi}\cos\alpha\eta).
\end{aligned}$$

This expression becomes

$$0 = \tan^2\theta \left[\begin{array}{l}
-A\,\varpi_{\beta\eta}(\varpi_{\alpha\xi}\cos\beta\eta - \varpi_{\alpha\eta}\cos\beta\xi) \\
+B\,\varpi_{\beta\xi}(\varpi_{\alpha\xi}\cos\beta\eta - \varpi_{\alpha\eta}\cos\beta\xi) \\
-C\,\varpi_{\beta\eta}(\varpi_{\beta\eta}\cos\beta\xi - \varpi_{\beta\xi}\cos\beta\eta) \\
+D\,\varpi_{\beta\xi}(\varpi_{\beta\eta}\cos\beta\xi - \varpi_{\beta\xi}\cos\beta\eta)
\end{array}\right]$$

$$\tan\theta \begin{bmatrix} -A\,\varpi_{\beta\eta}(\varpi_{\alpha\xi}\cos\alpha\eta - \varpi_{\alpha\eta}\cos\alpha\xi) \\ -A\,\varpi_{\alpha\eta}(\varpi_{\alpha\xi}\cos\beta\eta - \varpi_{\alpha\eta}\cos\beta\xi) \\ +B\,\varpi_{\alpha\xi}(\varpi_{\alpha\xi}\cos\beta\eta - \varpi_{\alpha\eta}\cos\beta\xi) \\ +B\,\varpi_{\beta\xi}(\varpi_{\alpha\xi}\cos\alpha\eta - \varpi_{\alpha\eta}\cos\alpha\xi) \\ -C\,\varpi_{\alpha\eta}(\varpi_{\beta\eta}\cos\beta\xi - \varpi_{\beta\xi}\cos\beta\eta) \\ -C\,\varpi_{\beta\eta}(\varpi_{\beta\eta}\cos\alpha\xi - \varpi_{\beta\xi}\cos\alpha\eta) \\ +D\,\varpi_{\beta\xi}(\varpi_{\beta\eta}\cos\alpha\xi - \varpi_{\beta\xi}\cos\alpha\eta) \\ +D\,\varpi_{\alpha\xi}(\varpi_{\beta\eta}\cos\beta\xi - \varpi_{\beta\xi}\cos\beta\eta) \end{bmatrix}$$

$$+ \begin{bmatrix} -A\,\varpi_{\alpha\eta}(\varpi_{\alpha\xi}\cos\alpha\eta - \varpi_{\alpha\eta}\cos\alpha\xi) \\ +B\,\varpi_{\alpha\xi}(\varpi_{\alpha\xi}\cos\alpha\eta - \varpi_{\alpha\eta}\cos\alpha\xi) \\ -C\,\varpi_{\alpha\eta}(\varpi_{\beta\eta}\cos\alpha\xi - \varpi_{\beta\xi}\cos\alpha\eta) \\ +D\,\varpi_{\alpha\xi}(\varpi_{\beta\xi}\cos\alpha\xi - \varpi_{\beta\xi}\cos\alpha\eta) \end{bmatrix}$$

For this to be true, the coefficient of $\tan\theta$ must vanish, while $\tan^2\theta = -\frac{p_\alpha}{p_\beta}$, for $p_\alpha\cos^2\theta + p_\beta\cos^2\theta = 0$; whence we have

$$0 = p_\alpha \begin{bmatrix} (A\,\varpi_{\beta\eta} - B\,\varpi_{\beta\xi})(\varpi_{\alpha\xi}\cos\beta\eta - \varpi_{\alpha\eta}\cos\beta\xi) \\ +(C\,\varpi_{\beta\eta} - D\,\varpi_{\beta\xi})(\varpi_{\beta\eta}\cos\beta\xi - \varpi_{\beta\xi}\cos\beta\eta) \end{bmatrix}$$

$$+ p_\beta \begin{bmatrix} +(B\,\varpi_{\alpha\xi} - A\,\varpi_{\alpha\eta})(\varpi_{\alpha\xi}\cos\alpha\eta - \varpi_{\alpha\eta}\cos\alpha\xi) \\ +(D\,\varpi_{\alpha\xi} - C\,\varpi_{\alpha\eta})(\varpi_{\beta\eta}\cos\alpha\xi - \varpi_{\beta\xi}\cos\alpha\eta) \end{bmatrix}$$

and also

$$\begin{aligned} 0 = &+(B\,\varpi_{\beta\xi} - A\,\varpi_{\beta\eta})(\varpi_{\alpha\xi}\cos\alpha\eta - \varpi_{\alpha\eta}\cos\alpha\xi) \\ &+(B\,\varpi_{\alpha\xi} - A\,\varpi_{\alpha\eta})(\varpi_{\alpha\xi}\cos\beta\eta - \varpi_{\alpha\eta}\cos\beta\xi) \\ &+(D\,\varpi_{\beta\xi} - C\,\varpi_{\beta\eta})(\varpi_{\beta\eta}\cos\alpha\xi - \varpi_{\beta\xi}\cos\alpha\eta) \\ &+(D\,\varpi_{\alpha\xi} - C\,\varpi_{\alpha\eta})(\varpi_{\beta\eta}\cos\beta\xi - \varpi_{\beta\xi}\cos\beta\eta). \end{aligned}$$

Thus we have four equations from which A, B, C, D can be eliminated, and the result is easily shown to be the homographic equation already obtained.

We have thus obtained the homographic equation from the fact that the two screws of zero pitch, on the instantaneous cylindroid, must be perpendicular to their corresponding impulsive screws. This gives two pairs of correspondents, and a pair of cross reciprocals give a third. This defines the homography.

V.—Investigation of the Reciprocal Correspondents.

Let us now take the case when two correspondents are reciprocal. We then have the homographic equation, and also

$$\varpi_{\alpha\eta}\cos\theta\cos\phi + \varpi_{\beta\xi}\sin\theta\sin\phi + \varpi_{\beta\eta}\sin\theta\cos\phi + \varpi_{\alpha\xi}\cos\theta\sin\phi = 0.$$

These determine θ and ϕ. Of course these must be imaginary, but it may be instructive to find them.

$$A\sin\theta\sin\phi + B\sin\theta\cos\phi + C\cos\theta\sin\phi + D\cos\theta\cos\phi = 0;$$

$$\tan\theta = -\frac{C\sin\phi + D\cos\phi}{A\sin\phi + B\cos\phi} = -\frac{C\tan\phi + D}{A\tan\phi + B};$$

$$\tan\theta = -\frac{\varpi_{\alpha\xi}\tan\phi + \varpi_{\alpha\eta}}{\varpi_{\beta\xi}\tan\phi + \varpi_{\beta\eta}};$$

whence

$$(C\tan\phi + D)(\varpi_{\beta\xi}\tan\phi + \varpi_{\beta\eta}) - (A\tan\phi + B)(\varpi_{\alpha\xi}\tan\phi + \varpi_{\alpha\eta}) = 0;$$

or

$$\tan^2\phi(C\varpi_{\beta\xi} - A\varpi_{\alpha\xi}) + \tan\phi(C\varpi_{\beta\eta} + D\varpi_{\beta\xi} - A\varpi_{\alpha\eta} - B\varpi_{\alpha\xi}) + D\varpi_{\beta\eta} - B\varpi_{\alpha\eta} = 0.$$

$$\begin{aligned}
C\varpi_{\beta\xi} - A\varpi_{\alpha\xi} &= p_\alpha\cos\beta\xi(\varpi_{\alpha\eta}\varpi_{\beta\xi}\cos\beta\xi - \varpi_{\alpha\xi}\varpi_{\beta\xi}\cos\beta\eta) \\
&\quad + p_\beta\cos\alpha\xi(\varpi_{\alpha\eta}\varpi_{\beta\xi}\cos\alpha\xi - \varpi_{\alpha\xi}\varpi_{\beta\xi}\cos\alpha\eta) \\
&\quad - p_\alpha\varpi^2_{\beta\xi}(\cos\alpha\xi\cos\beta\eta - \cos\alpha\eta\cos\beta\xi) \\
&\quad - p_\alpha\cos\beta\xi(\varpi_{\beta\eta}\varpi_{\alpha\xi}\cos\beta\xi - \varpi_{\beta\xi}\varpi_{\alpha\xi}\cos\beta\eta) \\
&\quad - p_\beta\cos\alpha\xi(\varpi_{\alpha\xi}\varpi_{\beta\eta}\cos\alpha\xi - \varpi_{\alpha\xi}\varpi_{\beta\eta}\cos\alpha\eta) \\
&\quad - p_\beta\varpi^2_{\alpha\xi}(\cos\alpha\xi\cos\beta\eta - \cos\alpha\eta\cos\beta\xi) \\
&= p_\alpha\cos^2\beta\xi(\varpi_{\alpha\eta}\varpi_{\beta\xi} - \varpi_{\beta\eta}\varpi_{\alpha\xi}) + p_\beta\cos^2\alpha\xi(\varpi_{\alpha\eta}\varpi_{\beta\xi} - \varpi_{\alpha\xi}\varpi_{\beta\eta}) \\
&\quad - (p_\alpha\varpi^2_{\beta\xi} + p_\beta\varpi^2_{\alpha\xi})(\cos\alpha\xi\cos\beta\eta - \cos\alpha\eta\cos\beta\xi) \\
&= (\varpi_{\alpha\eta}\varpi_{\beta\xi} - \varpi_{\beta\eta}\varpi_{\alpha\xi})(p_\alpha\cos^2\beta\xi + p_\beta\cos^2\alpha\xi) \\
&\quad + (\cos\alpha\eta\cos\beta\xi - \cos\beta\eta\cos\alpha\xi)(p_\alpha\varpi^2_{\beta\xi} + p_\beta\varpi^2_{\alpha\xi}).
\end{aligned}$$

$$\begin{aligned}
\varpi_{\beta\eta} D - \varpi_{\alpha\eta} B = \\
&+ p_\alpha \varpi_{\beta\eta} \varpi_{\alpha\eta} \cos\beta\eta \cos\beta\xi - p_\alpha \varpi_{\beta\eta} \varpi_{\alpha\xi} \cos^2\beta\eta \\
&+ p_\beta \varpi_{\beta\eta} \varpi_{\alpha\eta} \cos\alpha\xi \cos\alpha\eta - p_\beta \varpi_{\beta\eta} \varpi_{\alpha\xi} \cos^2\alpha\eta \\
&- p_\alpha \varpi_{\beta\eta} \varpi_{\beta\eta} \cos\alpha\xi \cos\beta\eta + p_\alpha \varpi_{\beta\eta} \varpi_{\beta\eta} \cos\alpha\eta \cos\beta\xi \\
&- p_\beta \varpi_{\alpha\eta} \varpi_{\beta\eta} \cos\alpha\eta \cos\alpha\xi + p_\beta \varpi_{\alpha\eta} \varpi_{\beta\xi} \cos\alpha\eta \cos\alpha\eta \\
&- p_\alpha \varpi_{\alpha\eta} \varpi_{\beta\eta} \cos\beta\eta \cos\beta\xi + p_\alpha \varpi_{\alpha\eta} \varpi_{\beta\xi} \cos\beta\eta \cos\beta\eta \\
&- p_\beta \varpi_{\alpha\eta} \varpi_{\alpha\eta} \cos\alpha\xi \cos\beta\eta + p_\beta \varpi_{\alpha\eta} \varpi_{\alpha\eta} \cos\alpha\eta \cos\beta\xi ;
\end{aligned}$$

which reduces to

$$\begin{aligned}
&(\varpi_{\alpha\eta} \varpi_{\beta\xi} - \varpi_{\alpha\xi} \varpi_{\beta\eta})(p_\beta \cos^2\alpha\eta + p_\alpha \cos^2\beta\eta) \\
&+ (\cos\alpha\eta \cos\beta\xi - \cos\alpha\xi \cos\beta\eta)(p_\alpha \varpi^2_{\beta\eta} + p_\beta \varpi^2 \alpha\eta).
\end{aligned}$$

We next calculate

$$\begin{aligned}
C\varpi_{\beta\eta} + D\varpi_{\beta\xi} - A\varpi_{\alpha\eta} - B\varpi_{\alpha\xi} = \\
&+ p_\alpha (\varpi_{\beta\xi} \varpi_{\alpha\eta} - \varpi_{\alpha\xi} \varpi_{\beta\eta}) \cos\beta\eta \cos\beta\xi \\
&+ p_\alpha (\varpi_{\alpha\eta} \varpi_{\beta\xi} - \varpi_{\alpha\xi} \varpi_{\beta\eta}) \cos\beta\eta \cos\beta\xi \\
&+ 2p_\beta (\varpi_{\beta\xi} \varpi_{\alpha\eta} - \varpi_{\beta\eta} \varpi_{\alpha\xi}) \cos\alpha\xi \cos\alpha\eta \\
&+ \cos\alpha\xi \cos\beta\eta (- p_\alpha \varpi_{\beta\eta} \varpi_{\beta\xi} - p_\alpha \varpi_{\beta\xi} \varpi_{\beta\eta} - p_\beta \varpi_{\alpha\eta} \varpi_{\alpha\xi} - p_\beta \varpi_{\alpha\eta} \varpi_{\alpha\xi}) \\
&+ \cos\alpha\eta \cos\beta\xi (+ p_\alpha \varpi_{\beta\eta} \varpi_{\beta\xi} + p_\alpha \varpi_{\beta\eta} \varpi_{\beta\xi} + 2p_\beta \varpi_{\alpha\eta} \varpi_{\alpha\xi}) ;
\end{aligned}$$

which reduces to

$$\begin{aligned}
&+ 2(\varpi_{\alpha\eta} \varpi_{\beta\xi} - \varpi_{\alpha\xi} \varpi_{\beta\eta})(p_\alpha \cos\beta\eta \cos\beta\xi + p_\beta \cos\alpha\eta \cos\alpha\xi) \\
&+ 2(\cos\alpha\eta \cos\beta\xi - \cos\alpha\xi \cos\beta\eta)(p_\alpha \varpi_{\beta\eta} \varpi_{\beta\xi} + p_\beta \varpi_{\alpha\eta} \varpi_{\alpha\xi}).
\end{aligned}$$

Thus the equation for $\tan\phi$ is

$$\begin{aligned}
&\tan^2\phi \left[\begin{matrix} (\varpi_{\alpha\eta} \varpi_{\beta\xi} - \varpi_{\beta\eta} \varpi_{\alpha\xi})(p_\alpha \cos^2\beta\xi + p_\beta \cos^2\alpha\xi) \\ + (\cos\alpha\eta \cos\beta\xi - \cos\beta\eta \cos\alpha\xi)(p_\alpha \varpi^2_{\beta\xi} + p_\beta \varpi^2_{\alpha\xi}) \end{matrix} \right] \\
&+ 2\tan\phi \left[\begin{matrix} (\varpi_{\alpha\eta} \varpi_{\beta\xi} - \varpi_{\alpha\xi} \varpi_{\beta\eta})(p_\alpha \cos\beta\eta \cos\beta\xi + p_\beta \cos\alpha\eta \cos\alpha\xi) \\ + (\cos\alpha\eta \cos\beta\xi - \cos\alpha\xi \cos\beta\eta)(p_\alpha \varpi_{\beta\eta} \varpi_{\beta\xi} + p_\beta \varpi_{\alpha\eta} \varpi_{\alpha\xi}) \end{matrix} \right] \\
&+ (\varpi_{\alpha\eta} \varpi_{\beta\xi} - \varpi_{\alpha\xi} \varpi_{\beta\eta})(p_\beta \cos^2\alpha\eta + p_\alpha \cos^2\beta\eta) \\
&+ (\cos\alpha\eta \cos\beta\xi - \cos\alpha\xi \cos\beta\eta)(p_\alpha \varpi^2_{\beta\eta} + p_\beta \varpi^2_{\alpha\eta}) = 0.
\end{aligned}$$

It may be written thus

$$0 = (\varpi_{\alpha\eta}\varpi_{\beta\xi} - \varpi_{\beta\eta}\varpi_{\alpha\xi})\left[\begin{array}{l} \tan^2\phi\,(p_\alpha\cos^2\beta\xi + p_\beta\cos^2\alpha\xi) \\ +\,2\tan\phi\,(p_\alpha\cos\beta\eta\cos\beta\xi + p_\beta\cos\alpha\eta\cos\alpha\xi) \\ +\,p_\beta\cos^2\alpha\eta + p_\alpha\cos^2\beta\eta \end{array}\right]$$

$$+\,(\cos\alpha\eta\cos\beta\xi - \cos\alpha\xi\cos\beta\eta)\left[\begin{array}{l} \tan^2\phi\,(p_\alpha\varpi^2_{\beta\xi} + p_\beta\varpi^2_{\alpha\xi}) \\ +\,2\tan\phi\,(p_\alpha\varpi_{\beta\eta}\varpi_{\beta\xi} + p_\beta\varpi_{\alpha\eta}\varpi_{\alpha\xi}) \\ +\,p_\alpha\varpi^2_{\beta\eta} + p_\beta\varpi^2_{\alpha\eta} \end{array}\right].$$

We shall now investigate the corresponding relation for $\tan\theta$. We have

$$\tan\phi = -\frac{B\tan\theta + D}{A\tan\theta + C},$$

$$\tan\phi = -\frac{\varpi_{\beta\eta}\tan\theta + \varpi_{\alpha\eta}}{\varpi_{\beta\xi}\tan\theta + \varpi_{\alpha\xi}};$$

or
$$\frac{B\tan\theta + D}{A\tan\theta + C} = \frac{\varpi_{\beta\eta}\tan\theta + \varpi_{\alpha\eta}}{\varpi_{\beta\xi}\tan\theta + \varpi_{\alpha\xi}};$$

$$\tan^2\theta\,(B\varpi_{\beta\xi} - A\varpi_{\beta\eta})$$
$$+\,\tan\theta\,(B\varpi_{\alpha\xi} + D\varpi_{\beta\xi} - A\varpi_{\alpha\eta} - C\varpi_{\beta\eta})$$
$$+\,D\varpi_{\alpha\xi} - C\varpi_{\alpha\eta} = 0.$$

We must now compute the coefficients, and it is easily shown that the quadratic for $\tan\theta$ is

$$0 = +\tan^2\theta\left[\begin{array}{l} p_\beta\,(\varpi_{\alpha\eta}\varpi_{\beta\xi} - \varpi_{\alpha\xi}\varpi_{\beta\eta})\,(\cos\alpha\xi\cos\beta\eta - \cos\alpha\eta\cos\beta\xi) \\ -\,p_\alpha\,(\varpi_{\beta\eta}\cos\beta\xi - \varpi_{\beta\xi}\cos\beta\eta)^2 - p_\beta\,(\varpi_{\beta\eta}\cos\alpha\xi - \varpi_{\beta\xi}\cos\alpha\eta)^2 \end{array}\right]$$

$$+\,2\tan\theta\left[\begin{array}{l} +\,p_\alpha\,(\varpi_{\beta\eta}\cos\beta\xi - \varpi_{\beta\xi}\cos\beta\eta)\,(\varpi_{\alpha\xi}\cos\beta\eta - \varpi_{\alpha\eta}\cos\beta\xi) \\ +\,p_\beta\,(\varpi_{\beta\eta}\cos\alpha\xi - \varpi_{\beta\xi}\cos\alpha\eta)\,(\varpi_{\alpha\xi}\cos\alpha\eta - \varpi_{\alpha\eta}\cos\alpha\xi) \end{array}\right]$$

$$+\left[\begin{array}{l} p_\alpha\,(\varpi_{\alpha\eta}\varpi_{\beta\xi} - \varpi_{\beta\eta}\varpi_{\alpha\xi})\,(\cos\alpha\xi\cos\beta\eta - \cos\alpha\eta\cos\beta\xi) \\ -\,p_\alpha\,(-\varpi_{\alpha\eta}\cos\beta\xi + \varpi_{\alpha\xi}\cos\beta\eta)^2 - p_\beta\,(\varpi_{\alpha\xi}\cos\alpha\eta - \varpi_{\alpha\eta}\cos\alpha\xi)^2 \end{array}\right].$$

The special case of two normal cylindroids, *i.e.* cylindroids which each contain one screw reciprocal to all the screws of the other, is worthy of attention.

Let θ be the screw on $\alpha\beta$, reciprocal to all screws of $\eta\xi$; then

$$\Sigma\, p_1\, \eta_1 (\alpha_1 \cos\theta + \beta_1 \sin\theta) = 0\,; \quad \text{or,} \quad \cos\theta\, \varpi_{\eta\alpha} + \sin\theta\, \varpi_{\beta\eta} = 0\,;$$

$$\Sigma\, p_1\, \xi_1 (\alpha_1 \cos\theta + \beta_1 \sin\theta) = 0\,; \quad \text{or,} \quad \cos\theta\, \varpi_{\alpha\xi} + \sin\theta\, \varpi_{\beta\xi} = 0\,;$$

which, remembering that the equations are homogeneous, may be written

$$\cos\theta = \varpi_{\beta\eta}, \qquad \sin\theta = -\varpi_{\eta\alpha}\,,$$

$$\cos\theta = \varpi_{\beta\xi}, \qquad \sin\theta = -\varpi_{\alpha\xi}\,;$$

we can write the homographic equation thus—

$$\begin{aligned}
0 = {} & \cos\phi\, p_\alpha \cos\beta\eta \cos\beta\xi\, (\varpi_{\alpha\eta}\cos\theta + \varpi_{\beta\eta}\sin\theta) \\
& \qquad - p_\alpha \cos^2\beta\eta \cos\phi\, (\varpi_{\alpha\xi}\cos\theta + \varpi_{\beta\xi}\sin\theta) \\
& + \cos\phi\, p_\beta \cos\alpha\xi \cos\alpha\eta\, (\varpi_{\alpha\eta}\cos\theta + \varpi_{\beta\eta}\sin\theta) \\
& \qquad - p_\beta \cos^2\alpha\eta \cos\phi\, (\varpi_{\alpha\xi}\cos\theta + \varpi_{\beta\xi}\sin\theta) \\
& + \sin\phi\, p_\alpha \cos^2\beta\xi\, (\varpi_{\alpha\eta}\cos\theta + \varpi_{\beta\eta}\sin\theta) \\
& \qquad - p_\alpha \sin\phi \cos\beta\xi \cos\beta\eta\, (\varpi_{\alpha\xi}\cos\theta + \varpi_{\beta\xi}\sin\theta) \\
& + \sin\phi\, p_\beta \cos^2\alpha\xi\, (\varpi_{\alpha\eta}\cos\theta + \varpi_{\beta\eta}\sin\theta) \\
& \qquad - p_\beta \sin\phi \cos\alpha\xi \cos\alpha\eta\, (\varpi_{\alpha\xi}\cos\theta + \varpi_{\beta\xi}\sin\theta) \\
& + (\cos\alpha\xi\cos\beta\eta - \cos\alpha\eta\cos\beta\xi) \\
& \qquad \left(\begin{array}{l} - p_\alpha\, \varpi_{\beta\eta}\cos\theta\cos\phi - p_\alpha\, \varpi_{\beta\xi}\cos\theta\sin\phi \\ \qquad + p_\beta\, \varpi_{\alpha\eta}\sin\theta\cos\phi + p_\beta\, \varpi_{\alpha\xi}\sin\theta\sin\phi \end{array}\right).
\end{aligned}$$

In the present case, the eight first lines vanish, and we have

$$\begin{aligned}
0 = {} & - p_\alpha\, \varpi_{\beta\eta}\cos\theta\cos\phi - p_\alpha\, \varpi_{\beta\xi}\cos\theta\sin\phi \\
& \qquad + p_\beta\, \varpi_{\alpha\eta}\sin\theta\cos\phi + p_\beta\, \varpi_{\alpha\xi}\sin\theta\sin\phi,
\end{aligned}$$

or $$0 = -p_\alpha \cos\theta\, (\varpi_{\beta\eta}\cos\phi + \varpi_{\beta\xi}\sin\phi) + p_\beta \sin\theta\, (\varpi_{\alpha\eta}\cos\phi + \varpi_{\alpha\xi}\sin\phi)\,;$$

whence ϕ must be the normal screw on $\eta\xi$.

In other words, if the two cylindroids are normal, then the two normal screws are correspondents.

VI.—The Rigid Body is defined when three pairs of correspondents are determined.

Let $\alpha\eta$, $\beta\xi$, $\gamma\zeta$ be three pairs of impulsive and instantaneous screws; let θ, ϕ be another pair. Then, if we denote by $L_{\alpha\beta} = 0$, and $M_{\alpha\beta} = 0$, the two fundamental equations

$$\frac{p_\alpha}{\cos\alpha\eta}\cos\beta\eta + \frac{p_\beta}{\cos\beta\xi}\cos\alpha\xi = 2\varpi_{\alpha\beta},$$

$$\frac{p_\alpha}{\cos\alpha\eta}\varpi_{\beta\eta} = \frac{p_\beta}{\cos\beta\xi},$$

we shall obtain six equations of the type

$$L_{\theta\alpha} = 0, \quad L_{\theta\beta} = 0, \quad L_{\theta\gamma} = 0,$$
$$M_{\theta\alpha} = 0, \quad M_{\theta\beta} = 0, \quad M_{\theta\gamma} = 0.$$

From these six it might be thought that $\phi_1, \ldots \phi_6$ could be eliminated, and thus it would, at first sight, seem that there must be an equation for θ to satisfy. It is, however, obvious that there can be no such condition, for θ can of course be chosen arbitrarily. The fact is, that these equations have a peculiar character which precludes the ordinary algebraical inference.

Since $\alpha\eta$, $\beta\xi$, $\gamma\zeta$ are three pairs of screws, fulfilling the necessary six conditions, a rigid body can be adjusted to them so that they are respectively impulsive and instantaneous. We take the six principal screws of inertia of this body as the screws of reference. We thus have, where ρ_α is a constant,

$$\rho_\alpha\eta_1 = a\alpha_1, \quad \rho_\alpha\eta_2 = -a\alpha_2, \quad \rho_\alpha\eta_3 = b\alpha_3, \quad \rho_\alpha\eta_4 = -b\alpha_4,$$
$$\rho_\beta\xi_1 = a\beta_1, \quad \ldots\ldots, \quad \ldots\ldots, \quad \ldots\ldots,$$
$$\rho_\gamma\zeta_1 = a\gamma_1, \quad \ldots\ldots, \quad \ldots\ldots, \quad \ldots\ldots.$$

By putting the coordinates in this form, we imply the fact that they satisfy the six equation of condition

$$L = 0, \qquad M = 0, \qquad \&c.$$

Substituting the coordinates in $L_{\theta\alpha} = 0$, we get

$$\begin{aligned}
&+ (\alpha_1 + \alpha_2)(\rho_\theta \phi_1 + \rho_\theta \phi_2) + (\alpha_3 + \alpha_4)(\rho_\theta \phi_3 + \rho_\theta \phi_4) \\
&+ (\alpha_5 + \alpha_6)(\rho_\theta \phi_5 + \rho_\theta \phi_6) \\
&+ (\theta_1 + \theta_2)(a\alpha_1 - a\alpha_2) + (\theta_3 + \theta_4)(b\alpha_3 - b\alpha_4) \\
&+ (\theta_5 + \theta_6)(c\alpha_5 - c\alpha_6) \\
&- 2(a\alpha_1\theta_1 - a\alpha_2\theta_2) - 2(b\alpha_3\theta_3 - b\alpha_4\theta_4) - 2(c\alpha_5\theta_5 - c\alpha_6\theta_6).
\end{aligned}$$

Let

$$\begin{aligned}
\rho_\theta \phi_1 - a\theta_1 &= X_1, & \rho_\theta \phi_2 + a\theta_2 &= X_2, \\
\rho_\theta \phi_3 - b\theta_3 &= X_3, & \rho_\theta \phi_4 + b\theta_4 &= X_4, \\
\rho_\theta \phi_5 - c\theta_5 &= X_5, & \rho_\theta \phi_6 + c\theta_6 &= X_6,
\end{aligned}$$

and the equation becomes

$$0 = (\alpha_1 + \alpha_2)(X_1 + X_2) + (\alpha_3 + \alpha_4)(X_3 + X_4) + (\alpha_5 + \alpha_6)(X_5 + X_6);$$

and the two other L equations give

$$\begin{aligned}
0 &= (\beta_1 + \beta_2)(X_1 + X_2) + (\beta_3 + \beta_4)(X_3 + X_4) + (\beta_5 + \beta_6)(X_5 + X_6), \\
0 &= (\gamma_1 + \gamma_2)(X_1 + X_2) + (\gamma_3 + \gamma_4)(X_3 + X_4) + (\gamma_5 + \gamma_6)(X_5 + X_6).
\end{aligned}$$

Now, if we were to eliminate $X_1 + X_2$, $X_3 + X_4$, $X_5 + X_6$ from these equations, we should have

$$0 = \begin{vmatrix} \alpha_1 + \alpha_2 & \alpha_3 + \alpha_4 & \alpha_5 + \alpha_6 \\ \beta_1 + \beta_2 & \beta_3 + \beta_4 & \beta_5 + \beta_6 \\ \gamma_1 + \gamma_2 & \gamma_3 + \gamma_4 & \gamma_5 + \gamma_6 \end{vmatrix}.$$

But this would only be the case if α, β, γ were parallel to a plane, which is not generally true. Therefore, we can only satisfy these equations, under ordinary circumstances, by the assumption

$$X_1 + X_2 = 0, \qquad X_3 + X_4 = 0, \qquad X_5 + X_6 = 0.$$

In like manner, the equations of the M type give

$$\begin{aligned}
0 &= \rho_\theta \varpi_{\alpha\phi} - \rho_\alpha \varpi_{\theta\eta}, \\
0 &= \rho_\theta \varpi_{\beta\phi} - \rho_\beta \varpi_{\theta\xi}, \\
0 &= \rho_\theta \varpi_{\gamma\phi} - \rho_\gamma \varpi_{\theta\xi}.
\end{aligned}$$

Substituting, in the first of these, we get

$$0 = +\rho_\theta(a\alpha_1\phi_1 - a\alpha_2\phi_2 + b\alpha_3\phi_3 - b\alpha_4\phi_4 + c\alpha_5\phi_5 - c\alpha_6\phi_6),$$
$$-\rho_\alpha(a\eta_1\theta_1 - a\eta_2\theta_2 + b\eta_3\theta_3 - b\eta_4\theta_4 + c\eta_5\theta_5 - c\eta_6\theta_6);$$

which reduces to

$$0 = a\alpha_1 X_1 - a\alpha_2 X_2 + b\alpha_3 X_3 - b\alpha_4 X_4 + c\alpha_5 X_5 - c\alpha_6 X_6;$$

but we have already seen that $X_1 + X_2 = 0$, &c., whence we obtain

$$0 = X_1(a\alpha_1 + a\alpha_2) + X_3(b\alpha_3 + b\alpha_4) + X_5(c\alpha_5 + c\alpha_6);$$

with the similar equations

$$0 = X_1(a\beta_1 + a\beta_2) + X_3(b\beta_3 + b\beta_4) + X_5(c\beta_5 + c\beta_6),$$
$$0 = X_1(a\gamma_1 + a\gamma_2) + X_3(b\gamma_3 + b\gamma_4) + X_5(c\gamma_5 + c\gamma_6).$$

These prove that, unless α, β, γ be parallel to a plane, we must have $X_1 = 0$, $X_3 = 0$, $X_5 = 0$. Combining these conditions with the last, we draw the general conclusion that

$$X_1 = 0, \quad X_2 = 0, \quad X_3 = 0, \quad X_4 = 0, \quad X_5 = 0, \quad X_6 = 0;$$

or $\rho_\theta\phi_1 = a\theta_1$; $\rho_\theta\phi_2 = -a\theta_2$; $\rho_\theta\phi_3 = b\theta_3$; &c.

Thus we demonstrate the important truth, that if a pair of screws θ, ϕ satisfy the six conditions, L and M, they stand to each other in the relation of impulsive screws and instantaneous screws.

Three pairs $a\eta$, $\beta\xi$, $\gamma\zeta$ fulfil six equations: if, therefore, α, β, γ be arbitrary, η, ξ, ζ will have still nine disposable quantities left. Thus, it would seem, that the problem of fitting the rigid body to the three pairs must be a definite one, as nine constants will fix the rigid body, except in so far as its density is concerned, which is not here significant. It remains, however, to be seen whether there can be more than one body with respect to which these three pairs are, respectively, impulsive screws and instantaneous screws. The answer is negative.

Suppose that the family of rigid bodies be found which make α, η and β, ξ impulsive and instantaneous. Let there be any third screw, γ, that

will be the locus of its impulsive screw, ζ, for all the different rigid bodies of the family.

ζ must satisfy the four equations

$$\frac{p_\alpha}{\cos \alpha\eta} \cos \gamma\eta + \frac{p_\gamma}{\cos \gamma\zeta} \cos \alpha\zeta = 2\varpi_{\alpha\gamma},$$

$$\frac{p_\beta}{\cos \beta\xi} \cos \gamma\xi + \frac{p_\gamma}{\cos \gamma\zeta} \cos \beta\zeta = 2\varpi_{\beta\gamma},$$

$$\frac{p_\alpha}{\cos \alpha\eta} \varpi_{\gamma\eta} = \frac{p_\gamma}{\cos \gamma\zeta} \varpi_{\alpha\zeta},$$

$$\frac{p_\beta}{\cos \beta\xi} \varpi_{\gamma\xi} = \frac{p_\gamma}{\cos \gamma\zeta} \varpi_{\beta\zeta}.$$

As there are four linear equations in the coordinates of ζ, we have the following theorem.

If α, η and β, ξ be given pairs of impulsive and instantaneous screws, then the locus of ζ, the impulsive screw corresponding to γ, as an instantaneous screw, is a cylindroid.

But this cylindroid is of a special type. It is indeed a plane surface rather than a cubic. The equations for ζ can have this form

$$\cos \alpha\zeta = A \cos \gamma\zeta, \qquad \varpi_{\alpha\zeta} = C \cos \gamma\zeta,$$
$$\cos \beta\zeta = B \cos \gamma\zeta, \qquad \varpi_{\beta\zeta} = D \cos \gamma\zeta,$$

in which A, B, C, D are constants.

The fact that $\cos \alpha\zeta$ and $\cos \gamma\zeta$ have one fixed ratio, and $\cos \beta\zeta$ and $\cos \gamma\zeta$ another, shows that the direction of ζ is fixed. The cylindroidal locus of ζ, therefore, degenerates to a system of parallel lines.

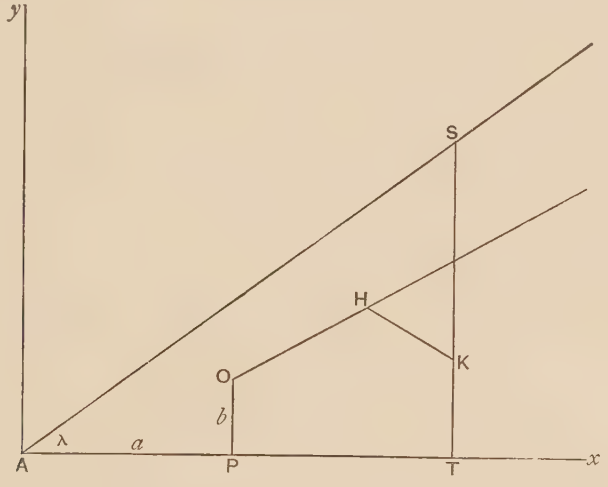
At first I was surprised to find that $\varpi_{\alpha\zeta}$ was a constant. But the necessity is soon apparent. If not constant, then there would have been some screw ζ, for which $\varpi_{\alpha\zeta}$ was 0. In this case, of course, $\varpi_{\gamma\eta}$ would be zero also. But γ and η being both given, this is of course not generally true. The only escape is for $\varpi_{\alpha\zeta}$ to be constant.

Let ST be one ϵ of the parallel system of screws, and let ST be its pitch. Then the degenerate cylindroid consists of all the lines parallel to ST in the plane, the pitch of each being the part between AT and AS.

Let AT be the axis of x,

Ay ,, ,, ,, y,

and a perpendicular to plane of paper, through A, the axis of z.



Let OH be any screw θ, of pitch p_θ, which makes angles α, β, γ with the axes of x, y, z.

It is required to find the virtual coefficient $\varpi_{\theta\epsilon}$ of θ with the screw ϵ on ST. We have

$$2\varpi_{\theta\epsilon} = (p_\theta + ST)\cos\beta - HK\sin\beta,$$

where HK is the common perpendicular to OH and ST.

The direction cosines of OH are $\cos\alpha$, $\cos\beta$, $\cos\gamma$.

Those of ϵ are 0, 1, 0.

Whence the direction cosines of HK are $-m\cos\gamma$, 0, $+m\cos\alpha$, where

$$1 = m^2\cos^2\gamma + m^2\cos^2\alpha = m^2\sin^2\beta;$$

$$\therefore \quad m = \pm\frac{1}{\sin\beta}.$$

The direction cosines of OH are, therefore,

$$\mp\frac{\cos\gamma}{\sin\beta}, \qquad 0, \qquad \pm\frac{\cos\alpha}{\sin\beta}.$$

From the polygon $POHKT$ we have, by resolving parallel to PT,

$$PT = OH\cos\alpha + \frac{\cos\gamma}{\sin\beta}HK.$$

Resolving parallel to Ay

$$OP + OH\cos\beta = KT.$$

Resolving parallel to Az

$$OH\cos\gamma = +HK\frac{\cos\alpha}{\sin\beta},$$

$$OH\cos\beta + b = TK;$$

$$PT\cos\gamma = OH\cos\alpha\cos\gamma + \frac{\cos^2\gamma}{\sin\beta}HK$$

$$= \left(\frac{\cos^2\alpha}{\sin\beta} + \frac{\cos^2\gamma}{\sin\beta}\right)HK = \sin\beta\, HK.$$

Whence

$$2\varpi_{\theta\epsilon} = p_\theta\cos\beta + ST\cos\beta - PT\cos\gamma,$$

$$ST = AT\tan\lambda = a\tan\lambda + PT\tan\lambda,$$

$$2\varpi_{\theta\epsilon} = p_\theta\cos\beta + (a\tan\lambda + PT\tan\lambda)\cos\beta - PT\cos\gamma,$$

$$= p_\theta\cos\beta + a\tan\lambda\cos\beta + PT(\tan\lambda\cos\beta - \cos\gamma);$$

whence we see the important result—

If $\tan\lambda\cos\beta - \cos\gamma = 0$, then $\varpi_{\theta\epsilon}$ is constant for all screws ϵ of the cylindroid. We have also

$$\cos\beta = \cos\lambda\sin\alpha,$$

$$\cos\gamma = \sin\lambda\sin\alpha,$$

$$\tan\lambda = \frac{\cos\gamma}{\cos\beta}.$$

Hence a plane through x and θ is inclined to xy at the angle λ.

This proves that the screw θ must, as the necessary and sufficient condition of having a constant virtual coefficient, be parallel to the plane

$$y \tan \lambda - z = 0.$$

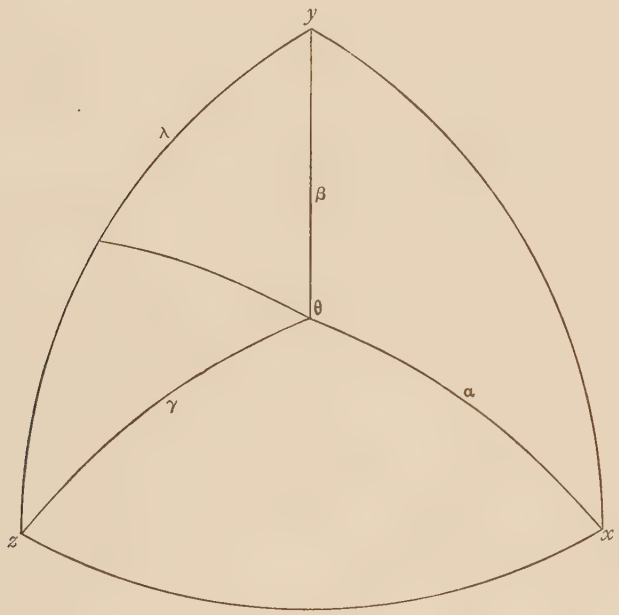
If ϵ passes through O, then $PT = 0$. The pitch of the screw through $O = a \tan \lambda$; whence

$$2 \varpi_{\theta\epsilon} = (p_\theta + a \tan \lambda) \cos \beta.$$

But we already found the expression

$$p_\theta \cos \beta + a \tan \lambda \cos \beta + PT(\tan \lambda \cos \beta - \cos \gamma),$$

which coincides with the above when $PT = 0$.



If ST be indefinitely remote, O coincides with A, and we have again the formula

$$x \cos \gamma = d \sin \beta;$$

whence the value of $2 \varpi_{\theta\epsilon}$ is $PT(\tan \lambda \cos \beta - \cos \gamma)$.

We have also

$$OH = \frac{PT\cos\alpha}{\sin^2\beta},$$

$$HK = \frac{PT\cos\gamma}{\sin\beta},$$

$$TK - OP = PT\frac{\cos\alpha\cos\beta}{\sin^2\beta}.$$

The last equation shows that the locus of K is a straight line.

I here investigate the same subject in a different manner. I take

as screws of reference six screws along the rectangular axes of pitches $+a$, $-a$, $+b$, $-b$, $+c$, $-c$, respectively.

In the plane of xy, at the point x, on axis of x, draw a line of length p, parallel to y. If θ be a screw of pitch p on this line, it is required to find the coordinates of θ. Let these coordinates be

$$\theta_1, \quad \theta_2, \quad \theta_3, \quad \theta_4, \quad \theta_5, \quad \theta_6;$$

the direction cosines of the screw, with respect to the three axes, are

$$\theta_1 + \theta_2, \quad \theta_3 + \theta_4, \quad \theta_5 + \theta_6;$$

since θ is parallel to y, we have,

$$\theta_1 + \theta_2 = 0, \quad \theta_3 + \theta_4 = 1, \quad \theta_5 + \theta_6 = 0;$$

so that the coordinates are

$$\theta_1, \quad -\theta_1, \quad \theta_3, \quad 1 - \theta_3, \quad \theta_5, \quad -\theta_5.$$

But these are, of course, far too general; they include every screw parallel to y.

The screw θ, obviously, has for its virtual coefficient, with the screw of pitch b on y,

$$\tfrac{1}{2}(b+p).$$

The coordinates of this screw are

$$0, \quad 0, \quad 1, \quad 0, \quad 0, \quad 0;$$

and the ordinary formula, for virtual coefficients, gives

$$b\theta_3 = \tfrac{1}{2}(b+p);$$

whence

$$\theta_3 = \frac{b+p}{2b};$$

θ is reciprocal to screw of pitch a on x. The coordinates of this screw are

$$1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0;$$

whence

$$\theta_1 = 0.$$

Finally, the figure shows, bearing in mind the general expression for the virtual coefficient,

$$\tfrac{1}{2}[(p_\alpha + p_\beta)\cos\alpha\beta - d\sin\alpha\beta];$$

that the virtual coefficient of θ, with screw on z, of pitch c, is simply

$$-\tfrac{1}{2}x;$$

whence, observing that the coordinates of this screw are

$$0, \quad 0, \quad 0, \quad 0, \quad 1, \quad 0,$$

we get

$$-\tfrac{1}{2}x = c\theta_5;$$

or,

$$\theta_5 = -\frac{x}{2c}.$$

Thus the coordinates required for θ are,

$$0, \quad 0, \quad \frac{b+p}{2b}, \quad \frac{b-p}{2b}, \quad \frac{-x}{2c}, \quad \frac{+x}{2c}.$$

These are complete; they are adequate for the coordinates of *any* screw which, lying in the plane of xy, is parallel to y, at the distance x, and has the pitch p.

This, it will be observed, is more general than a two-system; it is in fact a three-system. The two-system will now be developed.

Besides the screw θ, just defined by p and x, let there be another screw ϕ, of the same class, defined by p' and x'.

Then a twist velocity $\dot{\theta}$ on the first screw will have, for components,

$$0, \quad 0, \quad \dot{\theta}\frac{b+p}{2b}, \quad \dot{\theta}\frac{b-p}{2b}, \quad -\frac{x}{2c}\dot{\theta}, \quad +\frac{x}{2c}\dot{\theta};$$

and the twist velocity $\dot{\phi}$ on the second screw

$$0, \quad 0, \quad \dot{\phi}\frac{b+p'}{2b}, \quad \dot{\phi}\frac{b-p'}{2b}, \quad -\frac{x'}{2c}\dot{\phi}, \quad +\frac{x'}{2c}\dot{\phi}.$$

The result of these combined will have, as components, by simple addition,

$$0, \quad 0, \quad \tfrac{1}{2}(\dot{\theta}+\dot{\phi})+\frac{p\dot{\theta}+p'\dot{\phi}}{2b}, \quad \tfrac{1}{2}(\dot{\theta}+\dot{\phi})-\frac{p\dot{\theta}+p'\dot{\phi}}{2b},$$
$$-\frac{x\dot{\theta}+x'\dot{\phi}}{2c}, \quad +\frac{x\dot{\theta}+x'\dot{\phi}}{2c}.$$

These will be *proportional* to the coordinates of the resultant screw. To get the absolute coordinates we must multiply them by such a factor as will make them obey the fundamental law

$$(\theta_1+\theta_2)^2+(\theta_3+\theta_4)^2+(\theta_5+\theta_6)^2 = 1.$$

This factor clearly is $\dfrac{1}{\dot{\theta}+\dot{\phi}}$; whence the coordinates are

$$0, \quad 0, \quad \frac{1}{2}+\frac{p\dot{\theta}+p'\dot{\phi}}{2b(\dot{\theta}+\dot{\phi})}, \quad \frac{1}{2}-\frac{p\dot{\theta}+p'\dot{\phi}}{2b(\dot{\theta}+\dot{\phi})},$$
$$-\frac{1}{2c}\frac{x\dot{\theta}+x'\dot{\phi}}{\dot{\theta}+\dot{\phi}}, \quad +\frac{1}{2c}\frac{x\dot{\theta}+x'\dot{\phi}}{\dot{\theta}+\dot{\phi}}.$$

Hence we see that the resultant screw has a pitch

$$\frac{p\dot{\theta}+p'\dot{\phi}}{\dot{\theta}+\dot{\phi}},$$

and that it lies in the plane of xy, and parallel to y at a distance

$$\frac{x\dot{\theta}+x\dot{\phi}}{\dot{\theta}+\dot{\phi}}.$$

As the ratio of $\dot{\theta}$ to $\dot{\phi}$ varies, we have the collection of screws which form the degenerate cylindroid.

Let us now take some other screw, ω, with coordinates

$$\omega_1 , \qquad \omega_2 , \qquad . \quad . \quad . \quad . \quad . \qquad \omega_6 .$$

The virtual coefficient of this, with the screw just considered, is

$$b\,\omega_3\left(\frac{1}{2}+\frac{p\dot\theta+p'\dot\phi}{2b(\dot\theta+\dot\phi)}\right)-b\,\omega_4\left(\frac{1}{2}-\frac{p\dot\theta+p'\dot\phi}{2b(\dot\theta+\dot\phi)}\right)$$

$$-\frac{1}{2}\omega_5\frac{x\dot\theta+x'\dot\phi}{\dot\theta+\dot\phi}-\frac{1}{2}\omega_6\frac{x\dot\theta+x'\dot\phi}{\dot\theta+\dot\phi};$$

or,

$$\frac{b}{2}(\omega_3-\omega_4)+\frac{\omega_3+\omega_4}{2}\frac{p\dot\theta+p'\dot\phi}{\dot\theta+\dot\phi}-\frac{1}{2}(\omega_5+\omega_6)\frac{x\dot\theta+x'\dot\phi}{\dot\theta+\dot\phi};$$

which may be written

$$\frac{b}{2}(\omega_3-\omega_4)+\frac{\omega_3+\omega_4}{2}\frac{p\dot\theta+p'(\dot\theta+\dot\phi)-p'\dot\theta}{\dot\theta+\dot\phi}$$

$$-\frac{\omega_5+\omega_6}{2}\frac{x\dot\theta+x'(\dot\theta+\dot\phi-\dot\theta)}{\dot\theta+\dot\phi};$$

whence

$$\varpi=\frac{b}{2}(\omega_3-\omega_4)+\frac{p'}{2}(\omega_3+\omega_4)-\frac{x'}{2}(\omega_5+\omega_6)$$

$$+\frac{\dot\theta}{2(\dot\theta+\dot\phi)}\Big((\omega_3+\omega_4)(p-p')-(\omega_5+\omega_6)(x-x')\Big).$$

If the virtual coefficient is to be same for all the screws of the system, then the coefficient of $\frac{\dot\theta}{\dot\theta+\dot\phi}$ vanishes, and

$$(\omega_3+\omega_4)(p-p')-(\omega_5+\omega_6)(x-x')=0.$$

This is the necessary and the sufficient condition. It is, of course, the same as that already obtained

$$\tan\lambda=\frac{\cos\gamma}{\cos\beta}.$$

A difficulty may be here suggested.

Given α and η, β and ζ, and also γ, then the plane of ζ is determined from the equation on p. 157.

As $\varpi_{\alpha\zeta}$ and $\varpi_{\beta\zeta}$ are constant, both α and β must be parallel to the plane already considered. But as an impulsive screw could not be reciprocal to an instantaneous screw, it would seem that $\varpi_{\gamma\zeta}$ must *never* be zero. It must, therefore, apparently be constant, and accordingly ζ must be parallel to the same plane. Whence α, β, γ must be parallel to the same plane. But these three screws are quite arbitrary. Here then would seem to be a contradiction.

The difficulty can be explained as follows:—

Each rigid body, which conformed to the condition that α, β and η, ξ shall be two pairs of corresponding impulsive and instantaneous screws, will have a different screw ζ corresponding to a given screw γ. Thus, among the various screws ζ, in the degraded cylindroid, each will correspond to one rigid body. In general, of course, it would be impossible for ζ to be reciprocal to γ. It would be impossible for an impulsive wrench to make a body twist about a screw reciprocal thereto. Nevertheless, it seemed certain that, in general, there must be a screw ζ reciprocal to γ. For otherwise, α, β, γ should be all parallel to a plane, which, of course, is not generally true. If, however, either a, b, or c were zero, then the body will have *no mass*; consequently no impulse would be necessary to set it in motion. This clearly is the case when ζ is reciprocal to γ. We have thus got over the difficulty. ζ and γ being reciprocal, corresponds to the case when the rigid body is such that either a, b, or c is zero.

VII.—Additional Formulæ and concluding Notes.

The following point may be mentioned. I write the general formulæ

$$\frac{p_\alpha}{\cos\alpha\eta}\cos\gamma\eta + \frac{p_\gamma}{\cos\gamma\zeta}\cos\alpha\zeta = 2\varpi_{\alpha\gamma},$$

$$\frac{p_\beta}{\cos\beta\xi}\cos\gamma\xi + \frac{p_\gamma}{\cos\gamma\zeta}\cos\beta\zeta = 2\varpi_{\beta\gamma},$$

$$\frac{p_\alpha}{\cos\alpha\eta}\varpi_{\gamma\eta} = \frac{p_\gamma}{\cos\gamma\zeta}\varpi_{\alpha\zeta},$$

$$\frac{p_\beta}{\cos\beta\xi}\varpi_{\gamma\xi} = \frac{p_\gamma}{\cos\gamma\zeta}\varpi_{\beta\zeta}.$$

I take for α and β the two principal screws of inertia, with pitches $+a$, $-a$.

I suppose that the pitch of γ is p, and its direction angles, with respective, to the three principal axes are λ, μ, ν.

I suppose that the pitch of ζ is p', and its direction angles, with respective, to the three principal axes are λ', μ', ν'.

Let d be the shortest distance between the first principal axis and γ.

d' be the shortest distance between the first principal axis and ζ.

$$\frac{p_\alpha}{\cos\alpha\eta} = +a, \qquad \frac{p_\gamma}{\cos\gamma\zeta} = \omega, \qquad \cos\alpha\zeta = \cos\lambda',$$

$$\cos\gamma\eta = \cos\lambda, \qquad 2\varpi_{\alpha\gamma} = (a+p)\cos\lambda - d\sin\lambda,$$

$$\frac{p_\beta}{\cos\beta\xi} = -a, \qquad \cos\beta\zeta = \cos\lambda',$$

$$\cos\gamma\xi = \cos\lambda, \qquad 2\varpi_{\beta\gamma} = (-a+p)\cos\lambda - d\sin\lambda,$$

$$2\varpi_{\gamma\eta} = (a+p)\cos\lambda - d\sin\lambda, \qquad 2\varpi_{\alpha\zeta} = (a+p')\cos\lambda' - d'\sin\lambda',$$

$$2\varpi_{\gamma\xi} = (-a+p)\cos\lambda - d\sin\lambda, \qquad 2\varpi_{\beta\zeta} = (-a+p')\cos\lambda' - d'\sin\lambda';$$

with these substitutions, the general formulæ become

$$a\cos\lambda + \omega\cos\lambda' = (a+p)\cos\lambda - d\sin\lambda\,, \tag{i}$$

$$-a\cos\lambda + \omega\cos\lambda' = (-a+p)\cos\lambda - d\sin\lambda\,, \tag{ii}$$

$$+a((a+p)\cos\lambda - d\sin\lambda) = \omega((a+p')\cos\lambda' - d'\sin\lambda')\,, \tag{iii}$$

$$-a((-a+p)\cos\lambda - d\sin\lambda) = \omega((-a+p')\cos\lambda' - d'\sin\lambda'). \tag{iiii}$$

The two first equations give

$$\omega\cos\lambda' = p\cos\lambda - d\sin\lambda\,. \tag{v}$$

Adding the last two equations

$$a^2\cos\lambda = \omega(p'\cos\lambda' - d'\sin\lambda'). \tag{vi}$$

Subtracting the last equations (iii) and (iiii), we come on (v). Hence the four equations are equivalent to *two*, namely,

$$\omega\cos\lambda' = p\cos\lambda - d\sin\lambda\,,$$
$$a^2\cos\lambda = \omega(p'\cos\lambda' - d'\sin\lambda')\,.$$

If ϵ be the angle between γ and ζ,

$$\omega = \frac{p}{\cos\epsilon}\,,$$

$$p\cos\lambda' = \cos\epsilon(p\cos\lambda - d\sin\lambda)$$
$$a^2\cos\lambda\cos\epsilon = p(p'\cos\lambda' - d'\sin\lambda').$$

It may be convenient to have these results

$$d\sin\lambda = p\cos\lambda - p\cos\lambda'\sec\epsilon\,,$$

$$d'\sin\lambda' = p'\cos\lambda' - \frac{a^2}{p}\cos\lambda\cos\epsilon.$$

I will now write the formulæ which arise if we first take, as α and β, the two principal screws of inertia of pitches $+a$ and $+b$, and then the two principal screws of inertia of pitches $-a$ and $-b$.

$$\frac{p_\alpha}{\cos \alpha\eta} = +a, \quad \cos\gamma\eta = \cos\lambda, \quad \frac{p_\gamma}{\cos\gamma\zeta} = \frac{p}{\cos\epsilon}, \quad \cos\alpha\zeta = \cos\lambda',$$

$$\frac{p_\beta}{\cos\beta\zeta} = +b, \quad \cos\gamma\xi = \cos\mu, \quad \cos\beta\zeta = \cos\mu';$$

$$2\varpi_{\alpha\gamma} = (p+a)\cos\lambda - d_x \sin\lambda,$$

$$2\varpi_{\beta\gamma} = (p+b)\cos\mu - d_y \sin\mu,$$

$$2\varpi_{\gamma\eta} = (p+a)\cos\lambda - d_x \sin\lambda,$$

$$2\varpi_{\alpha\zeta} = (p'+a)\cos\lambda' - d'_x \sin\lambda',$$

$$2\varpi_{\gamma\xi} = (p+b)\cos\mu - d_y \sin\mu,$$

$$2\varpi_{\beta\zeta} = (p'+b)\cos\mu' - d'_y \sin\mu'.$$

The formulæ become, with these substitutions,

$$+a\cos\lambda + \frac{p}{\cos\epsilon}\cos\lambda' = p\cos\lambda + a\cos\lambda - d_x\sin\lambda,$$

$$+b\cos\mu + \frac{p}{\cos\epsilon}\cos\mu' = p\cos\mu + b\cos\mu - d_y\sin\mu,$$

$$(ap+a^2)\cos\lambda - a d_x \sin\lambda = \frac{p}{\cos\epsilon}(p'\cos\lambda' + a\cos\lambda' - d'_x\sin\lambda'),$$

$$(bp+b^2)\cos\mu - b d_y \sin\mu = \frac{p}{\cos\epsilon}(p'\cos\mu' + b\cos\mu' - d'_y\sin\mu');$$

or thus,

$$\frac{p}{\cos\epsilon}\cos\lambda' = p\cos\lambda - d_x\sin\lambda,$$

$$\frac{p}{\cos\epsilon}\cos\mu' = p\cos\mu - d_y\sin\mu,$$

$$a^2\cos\lambda - \frac{p}{\cos\epsilon}(p'\cos\lambda' - d'_x\sin\lambda') = a\left(-p\cos\lambda + d_x\sin\lambda + \frac{p}{\cos\epsilon}\cos\lambda'\right),$$

$$b^2\cos\mu - \frac{p}{\cos\epsilon}(p'\cos\mu' - d'_y\sin\mu') = b\left(-p\cos\mu + d_y\sin\mu + \frac{p}{\cos\epsilon}\cos\mu'\right);$$

whence the four equations reduce to

$$\frac{p}{\cos\epsilon}\cos\lambda' = p\cos\lambda - d_x\sin\lambda\,,$$

$$\frac{p}{\cos\epsilon}\cos\mu' = p\cos\mu - d_y\sin\mu\,,$$

$$a^2\cos\lambda = \frac{p}{\cos\epsilon}(p'\cos\lambda' - d'_x\sin\lambda')\,,$$

$$b^2\cos\mu = \frac{p}{\cos\epsilon}(p'\cos\mu' - d'_y\sin\mu')\,.$$

Note that these equations are unaltered if we make the pitches $-a$ and $-b$, instead of $+a$ and $+b$. Of course this is because the same relations hold for the principal screws of inertia with pitches $-a$ and $-b$.

We may write these otherwise

$$a^2\cos\lambda = p'(p\cos\lambda - d_x\sin\lambda) - \frac{p}{\cos\epsilon}\sin\lambda' d'_x\,,$$

$$b^2\cos\mu = p'(p\cos\mu - d_y\sin\mu) - \frac{p}{\cos\epsilon}d'_y\sin\mu';$$

whence

$$\frac{p}{\cos\epsilon}\sin\lambda' d'_x = (pp' - a^2)\cos\lambda - p' d_x\sin\lambda\,,$$

$$\frac{p}{\cos\epsilon}\sin\mu' d'_y = (pp' - b^2)\cos\mu - p' d_y\sin\mu\,.$$

We have also

$$d_x = p\cot\lambda - \frac{p}{\cos\epsilon}\frac{\cos\lambda'}{\sin\lambda}\,,$$

$$d_y = p\cot\mu - \frac{p}{\cos\epsilon}\frac{\cos\mu'}{\sin\mu}\,.$$

$$d'_x = p'\cot\lambda' - \frac{\cos\epsilon}{p}a^2\cos\lambda\,,$$

$$d'_y = p'\cot\mu' - \frac{\cos\epsilon}{p}b^2\cos\mu\,.$$

We may here obtain the coordinates of an impulsive screw and its corresponding instantaneous screw.

We have

$$\theta_1 = \frac{\varpi_{1\theta}}{p_1},$$

whence

$$\theta_1 = \frac{1}{2a}\Big((a+p)\cos\lambda - d_x \sin\lambda\Big)$$

$$= \frac{1}{2a}\left(a\cos\lambda + \frac{p}{\cos\epsilon}\cos\lambda'\right).$$

The six coordinates of θ are

$$\frac{\cos\lambda}{2} + \frac{p}{2a\cos\epsilon}\cos\lambda',$$

$$\frac{\cos\lambda}{2} - \frac{p}{2a\cos\epsilon}\cos\lambda',$$

$$\frac{\cos\mu}{2} + \frac{p}{2b\cos\epsilon}\cos\mu',$$

$$\frac{\cos\mu}{2} - \frac{p}{2b\cos\epsilon}\cos\mu',$$

$$\frac{\cos\nu}{2} + \frac{p}{2c\cos\epsilon}\cos\nu',$$

$$\frac{\cos\nu}{2} - \frac{p}{2c\cos\epsilon}\cos\nu'.$$

In like manner, the coordinates of the corresponding impulsive wrench are

$$\frac{\cos\lambda'}{2} + \frac{a}{2p\sec\epsilon}\cos\lambda, \qquad \frac{\cos\lambda'}{2} - \frac{a}{2p\sec\epsilon}\cos\lambda,$$

$$\frac{\cos\mu'}{2} + \frac{b}{2p\sec\epsilon}\cos\mu, \qquad \frac{\cos\mu'}{2} - \frac{b}{2p\sec\epsilon}\cos\mu,$$

$$\frac{\cos\nu'}{2} + \frac{c}{2p\sec\epsilon}\cos\nu, \qquad \frac{\cos\nu'}{2} - \frac{c}{2p\sec\epsilon}\cos\nu.$$

We check this by observing that these may be found from the instantaneous screw by the factors $+a\frac{\cos\epsilon}{p}$, $-a\frac{\cos\epsilon}{p}$, $b\frac{\cos\epsilon}{p}$, $-b\frac{\cos\epsilon}{p}$, &c.

The work can also be verified by forming the pitch of the instantaneous screw which reduces to p.

An interesting expression for p' appears: we get

$$p\,p' = \cos\epsilon\,(a^2\cos\lambda\cos\lambda' + b^2\cos\mu\cos\mu' + c^2\cos\nu\cos\nu')\,.$$

It is easy to verify this, for

$$a\cos\lambda = a(\theta_1+\theta_2) = \frac{p}{\cos\epsilon}(\phi_1-\phi_2),$$

$$\cos\lambda' = \phi_1+\phi_2,$$

$$a\cos\lambda\cos\lambda' = \frac{p}{\cos\epsilon}(\phi_1^2-\phi_2^2),$$

$$a^2\cos\lambda\cos\lambda' = \frac{p}{\cos\epsilon}(a\phi_1^2-a\phi_2^2);$$

whence the formula given above. We also find

$$\varpi_{\theta\phi} = \frac{1}{4}a\left[\begin{array}{l}(\cos\lambda+\dfrac{p}{a\cos\epsilon}\cos\lambda')(\cos\lambda'+\dfrac{a}{p\sec\epsilon}\cos\lambda)\\ -(\cos\lambda-\dfrac{p}{a\cos\epsilon}\cos\lambda')(\cos\lambda'-\dfrac{a}{p\sec\epsilon}\cos\lambda)\end{array}\right]$$

$$+ \text{ two similar terms}$$

$$= +\frac{p}{2\cos\epsilon}\cos^2\lambda' + \frac{a^2\cos\epsilon}{2p}\cos^2\lambda$$

$$+\frac{p}{2\cos\epsilon}\cos^2\mu' + \frac{\cos\epsilon}{2p}b^2\cos^2\mu$$

$$+\frac{p}{2\cos\epsilon}\cos^2\nu' + \frac{\cos\epsilon}{2p}c^2\cos^2\nu\,.$$

$$= \frac{p}{2\cos\epsilon} + \frac{\cos\epsilon}{2p}\left(a^2\cos^2\lambda + b^2\cos^2\mu + c^2\cos^2\nu\right)$$

$$= \frac{\cos\epsilon}{2p}\left[\frac{p^2}{\cos^2\epsilon} + a^2\cos^2\lambda + b^2\cos^2\mu + c^2\cos^2\nu\right].$$

This shows, as might have been expected, the essentially positive character of $\varpi_{\theta\phi}$.